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On general characterization of Young measures associated with Borel functions

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We prove that the Young measure associated with a Borel function f is a probability distribution of the random variable $f(U)$, where U has a uniform distribution on the domain of f. Examples of specific applications of the main result, including Monte Carlo method of simulation of Young measures, are presented with comments.

Key words: Young measures, microstructure, Borel functions, uniform distribution, Monte Carlo simulation

1. Introduction

One of the major difficulties in variational problems is minimization of functionals which are bounded from below but do not attain their infima. For example in nonlinear elasticity they are the energy functionals of certain laminates or elastic crystals, see e.g. [14–16]. Minimizing sequences of such functionals are bounded, divergent in norm while weakly* convergent. Elements of these sequences are functions rapidly oscillating around weak* limits. This behavior of the minimizing sequences can be looked at as a description of the structure of considered materials: it depicts a *microstructure* – a phenomenon that can be observed in considered materials under a microscope and is placed between micro- and macroscopic scale. It turns out that the spatial oscillatory properties of minimizing sequences are well described by the *Young measures* – objects introduced as 'generalized curves', by Laurence Chisholm Young in [22] and later as 'generalized surfaces' in [23].

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As one of the most efficient techniques, the direct method can be applied: there always exists a minimizing sequence for $\mathcal{J}$, that is a sequence (u_n), $u_n\colon \mathbb{R}^d \to \mathbb{R}^l$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \mathcal{J}(u_n) = \inf \mathcal{J}$. Additionally, if $\mathcal{J}$ is coercive, (u_n) is always bounded. However, if $\mathcal{J}$ does not attain its infimum then the elements of (u_n) are functions of highly oscillatory nature. Moreover, weak* convergence in L^∞ of (u_n) to some function u_0 does not guarantee, that the sequence $(\varphi(u_n))$ of compositions of u_n, even with continuous function φ, is weak* convergent in L^∞. Indeed, in general it is not convergent not only to $\varphi(u_0)$, but to any function with domain in $\mathbb{R}^d$: its 'generalized limit' is a Young measure. However, it is a very difficult task to compute the Young measure associated with particular minimizing sequence.

The mature form of Young's theorem has been proved by J.M. Ball in [3] (see also [15], theorem 6.2). According to these theorems, we say that under their assumptions the considered sequences 'generate' appropriate Young measures. This approach is studied for example in [15] in detail.

Alternatively, we can look at the Young measure as at object associated with any measurable function defined on a nonempty, open, bounded subset Ω of $\mathbb{R}^d$ with values in a compact subset K of $\mathbb{R}^l$. Such a conclusion can be derived from the Theorem 3.6 in [20]. Thank to this theorem it can be proved that the Young measure associated with a simple function is the convex combination of Dirac measures. These Dirac measures are concentrated at the values of the simple function under consideration while coefficients of the convex combination are proportional to the Lebesgue measure of the sets on which the respective values are taken on by the function; see [17] for details and more general result concerning simple method of obtaining explicit form of Young measures associated with oscillating functions. This method does not need advanced functional analytic methods, it is based on the change of variable theorem.

In this article we generalize the above results. We prove a theorem providing general yet simple description of Young measures associated with Borel functions. As a consequence, the theorem enables one to compute explicit formulae of probability density functions of the Young measures in many interesting cases. What is more, the formulation of the result makes it well-suited for applications.

Since Young measures are widely used in many areas of theoretical and applied sciences (see for example [2, 12–14, 16]), our result provide a handy tool of obtaining their explicit form.

The main theorem of this article states that the Young measure associated with any Borel function f defined on the set $\Omega \subset \mathbb{R}^d$ with positive Lebesgue measure M and values in a compact set $K \subset \mathbb{R}^l$, is in fact a probability distribution of a random variable $X = f(U)$, where U is uniformly distributed on Ω. It thus introduces probabilistic methods to the theory of Young measures. A main application of the theorem is the possibility of using Monte Carlo simulation in calculating explicit form of a Young measure associated with Borel function.

We illustrate the result with two one-dimensional examples based on this result. The first one describes calculating the explicit form of the density of the Young measure associated with certain discontinuous function. The second example illustrates generating Young measure with the Monte Carlo simulation.

2. An outline of the Young measures theory

Literature concerning Young measures is large and still growing. An interested reader may consult for example: [3, 5, 7, 8, 15, 20] for general theory; [1, 13, 20] for applications in differential equations and optimization; [12, 14–16] for engineering applications, in particular in nonlinear elasticity theory and micromagnetism; [4, 6] for numerical aspects of the use of Young measures in engineering and differential equations; [2] for applications in economical sciences, and the references cited there. As it has been mentioned earlier, our presentation follows [20]. In particular, theorems that plays crucial role in the sequel, namely Theorem 2, Theorem 3 and Corollary 1, are stated and proved there.

Let $\mathbb{R}^d \supset \Omega$ be nonempty, bounded open set and let $K \subset \mathbb{R}^l$ be compact. Let (f_n) be a sequence of functions from Ω to K, convergent to some function f_0 weakly* in L^∞. Finally, let φ be an arbitrary continuous real valued function on $\mathbb{R}^l$. Then the sequence $(\varphi(f_n))$ is uniformly bounded in L^∞ norm and therefore by the Banach – Alaoglu theorem there exists a subsequence of $(\varphi(f_n))$ weakly* convergent to some function g. In general $g \neq \varphi((f_0))$. L.C. Young proved in [22], that there exists a subsequence of $(\varphi(f_n))$, not relabelled, and a family $(\nu_x)_{x\in\Omega}$ of probability measures with supports $\operatorname{supp}\nu_x \subseteq K$, such that $\forall\, \varphi \in C(\mathbb{R}^l)$ $\forall w \in L^1(\Omega)$ there holds

$$\lim_{n\to\infty} \int\limits_{\Omega} \varphi(f_n(x))w(x)\,\mathrm{d}x = \int\limits_{\Omega}\int\limits_{K} \varphi(s)\nu_x(\mathrm{d}s)w(x)\,\mathrm{d}x := \int\limits_{\Omega} \overline{\varphi}(x)w(x)\,\mathrm{d}x.$$

This family of probability measures is today called a *Young measure associated with the sequence* (f_n).

The simplest, but by no means trivial and often appearing in applications, form of Young measure is a *homogeneous* Young measure. For example, Young measures associated with minimizing sequences of the energy functionals of laminates are homogeneous ones, see e.g. [14]. The homogeneous Young measure is a ‘family’ $(\nu_x)_{x\in\Omega}$ that does not depend on the variable x.

In 1989 J.M. Ball proved the following theorem. Let Ω be a measurable subset of $\mathbb{R}^d$, $\nu\colon\ [0,+\infty) \to [0,+\infty)$ a continuous, nondecreasing function such that $\lim\limits_{t\to\infty} \nu(t) = +\infty$. By ψ we denote a function $\psi\colon\ \Omega\times\mathbb{R}^l \ni (x,\lambda) \to \psi(x,\lambda) \in \overline{\mathbb{R}}$ satisfying Carathéodory conditions: it is measurable with respect to the first, and continuous with respect to the second variable. Consider further a sequence (f_n) of functions on Ω with values in $\mathbb{R}^l$, satisfying the condition

$$\sup_n \int\limits_\Omega \nu(|f_n(x)|)\,\mathrm{d}x < +\infty.$$

Theorem 1. [3] *Under the above assumptions, there exists a subsequence of* (f_n)*, not relabelled, and a family* $(\nu_x)_{x\in\Omega}$ *of probability measures, dependent measurably on* x*, such that if for any Carathéodory function* ψ *the sequence* $(\psi(x, f_n(x))$ *is weakly convergent in* $L^1(\Omega)$*, then its weak limit is a function*

$$\overline{\psi}(x) = \int\limits_{\mathbb{R}^l} \psi(x,\lambda)\,\mathrm{d}\nu_x(\lambda).$$

We now turn our attention to the presentation of the Young measures as in [20]. In general, Young measures can be looked at as the elements of the space conjugate to the space $L^1(\Omega, C(K))$ of Bochner integrable functions on $\Omega \subset \mathbb{R}^d$ with values in $C(K)$. Here Ω is bounded open set of positive Lebesgue measure M, while $K \subset \mathbb{R}^l$ is nonempty and compact. The space $L^1(\Omega, C(K))$ is isometrically isomorphic to the space $Car(\Omega, K;\mathbb{R})$ of the Carathéodory functions, equipped with the norm

$$\|h\|_{Car} := \int\limits_\Omega \sup_{k\in K} \big|h(x,k)\big|\,\mathrm{d}x.$$

Let $h \in L^1(\Omega, C(K))$. Denote by $\mathcal{U}$ the set of all measurable functions on Ω with values in K. Consider a mapping

$$i\colon\ \mathcal{U} \to L^1\big(\Omega, C(K)\big)^*$$

defined by the formula

$$\langle i(f), h\rangle := \int\limits_\Omega h(x, f(x))\,\mathrm{d}x.$$

By $Y(\Omega, K)$ we denote the weak* closure of the set $i(\mathcal{U})$ in $L^1(\Omega, C(K))^*$:

$$Y(\Omega, K) := \Big\{L^1(\Omega, C(K))^* \ni \eta :\ \exists (f_\alpha) \subset \mathcal{U} : i(f_\alpha) \xrightarrow[\alpha]{w^*} \eta\Big\}.$$

Denote by

- $rca(K)$ – the space of regular, countably additive scalar measures on K, equipped with the norm $\|m\|_{rca(K)} := |m|(\Omega)$, where $|\cdot|$ stands in this case for the total variation of the measure m. With this norm $rca(K)$ is a Banach space;
- $rca^1(K)$ – the subset of $rca(K)$ with elements being probability measures on K;
- $L^\infty_{w^*}(\Omega, \mathrm{rca}(K))$ – the set of the weakly* measurable mappings

  $$\nu \colon \Omega \ni x \to \nu(x) \in rca(K)$$

  and such that

  $$\operatorname{ess\,sup}\big\{\|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega\big\} < +\infty.$$

  We equip this set with the norm

  $$\|\nu\|_{L^\infty_{w^*}(\Omega,\mathrm{rca}(K))} := \operatorname{ess\,sup}\big\{\|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega\big\}.$$

  By the Dunford–Pettis theorem this space is isometrically isomorphic with the space $L^1(\Omega, C(K))^*$.

Now define an element η of $L^1(\Omega, C(K))^*$ by the formula

$$\eta \colon L^1(\Omega, C(K)) \ni h \to \langle \eta, h \rangle := \int_\Omega \left(\int_K h(x,k) \, \mathrm{d}\nu_x(k) \right) \mathrm{d}x,$$

which in turn will be the value of the mapping

$$\psi \colon L^\infty_{w^*}(\Omega, \mathrm{rca}(K)) \ni \nu \to \psi(\nu) := \eta \in L^1(\Omega, C(K))^*$$

Theorem 2. *The mapping ψ defined above is an isometric isomorphism between the spaces $L^\infty_{w^*}(\Omega, \mathrm{rca}(K))$ and $L^1(\Omega, C(K))^*$.*

The set of the Young measures on the compact set $K \subset \mathbb{R}^l$ will be denoted by $\mathcal{Y}(\Omega, K)$:

$$\mathcal{Y}(\Omega, K) := \big\{\nu = (\nu(x)) \in L^\infty_{w^*}(\Omega, \mathrm{rca}(K)) : \ \nu_x \in \mathrm{rca}^1(K) \text{ for a.a } x \in \Omega\big\}.$$

We will write ν_x or $(\nu_x)_{x\in\Omega}$ instead of $\nu(x)$.

Finally, we define the Dirac mapping δ: $\forall x \in \Omega$

$$\delta \colon \mathcal{U} \ni f \to \big[\delta(f)\big](x) := \delta_{f(x)} \in \mathcal{Y}(\Omega, K). \tag{1}$$

Theorem 3. *The diagram*

$$\begin{tikzcd} & \mathcal{U} \arrow[dl, "\delta"'] \arrow[dr, "i"] & \\ \mathcal{Y}(\Omega;K) \arrow[rr, leftrightarrow, "\psi"'] & & Y(\Omega;K) \end{tikzcd}$$

is commutative.

This means, that for any $f \in \mathcal{U}$ there exists a Young measure associated with it. It is seen, that the values of the weak* measurable mapping δ are not Young measures in their explicit form associated with arguments of δ. However, we can modify this function in a simple way such that the modified one assigns to a function a Young measure that is associated with this function (see [19]).

Corollary 1. *The set $\mathcal{Y}(\Omega;K)$ of all Young measures is a convex, compact, sequentially compact set in which $\delta(U)$ is dense.*

Denote by μ the normalized Lebesgue measure on Ω: $d\mu(x) := \frac{1}{M}dx$ with a d-dimensional Lebesgue measure dx. Let $\{\Omega\}_{i=1}^{n}$ be a partition of Ω into open, pairwise disjoint subsets Ω_i with Lebesgue measure $m_i > 0$, such that $\bigcup_{i=1}^{n} cl(\Omega_i) = cl(\Omega)$, where '$cl$' stands for 'closure'. By $\mathbf{1}_A$ we denote the characteristic function of the set A.

Theorem 4. *(see e.g. [17]) Choose and fix points $p_i \in \mathbb{R}^l$, $i = 1, 2, \ldots, n$, and let f be a simple function:*

$$f := \sum_{i=1}^{n} p_i \mathbf{1}_{\Omega_i}.$$

Then the Young measure associated with f is homogeneous and is of the form

$$\nu_x = \frac{1}{M} \sum_{i=1}^{n} m_i \delta_{p_i}.$$

The Young measure associated with a simple function will be called a *simple Young measure*.

Recall that a sequence (ν_n) of bounded measures on a compact set $K \subset \mathbb{R}^l$ converges weakly* to a measure ν_0, if $\forall \beta \in C(K, \mathbb{R})$ there holds

$$\lim_{n\to\infty} \int_K \beta(k)\, \mathrm{d}\nu_n(k) = \int_K \beta(k)\, \mathrm{d}\nu_0(k).$$

The fact that Borel functions are pointwise limits of the appropriate sequences of simple functions and the Corollary 1 imply the following.

Corollary 2. *The set of all simple Young measures is weak* dense in the set of the Young measures associated with functions from $\mathcal{U}$.*

3. Main result

To set up notation, we recall first standard probabilistic notions needed further.

3.1. Some necessary notions from probability theory and notation

If Σ is a σ-algebra of subsets of a nonempty set A and P – a measure on Σ, then the triple (A, Σ, P) is called a measure space, and a *probability space* if P is a probability measure. A *random variable* (or a *random vector*) $X\colon A \to \mathbb{R}^d$ is a function such that for any Borel set $B \subseteq \mathbb{R}^d$ there holds $X^{-1}(B) \in \Sigma$. If $\varphi\colon \mathbb{R}^d \to \mathbb{R}^l$ is a Borel function, then $\varphi(X)$ is a random variable. The *probability distribution* on $\mathbb{R}^d$ is any probability measure P on the σ-algebra $\mathcal{B}(\mathbb{R}^d)$ of Borel subsets of $\mathbb{R}^d$. The probability distribution of a random variable X with values in $\mathbb{R}^d$ is a probability measure P_X on $\mathbb{R}^d$ defined for any $B \in \mathcal{B}(\mathbb{R}^d)$ by the equality $P_X(B) := P(X^{-1}(B))$. Consequently, for the distribution of the random variable $\varphi(X)$ we have: for any $C \in \mathcal{B}(\mathbb{R}^l)$

$$P_{\varphi(X)}(C) = P(\varphi(X)^{-1}(C)) = P_X(\varphi^{-1}(C)).$$

If P is a probability distribution on $\mathbb{R}^d$ and for some Lebesgue integrable function $g\colon \mathbb{R}^d \to \mathbb{R}$ there holds: $\forall\, A \in \mathcal{B}(\mathbb{R}^d)\ P(A) = \int\limits_A g(x)\,\mathrm{d}x$, then the function g is called a *density* of P.

Let Ω be a Borel subset of $\mathbb{R}^d$ with Lebesgue measure $M > 0$. We say that random variable $U\colon \mathbb{R}^d \to \mathbb{R}^l$ is *uniform* on Ω, if its density g_u is of the form

$$g_U(x) = \begin{cases} \dfrac{1}{M}, & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}$$

The probability distribution P_U is then called the *uniform distribution*.

3.2. Young measures associated with bounded Borel functions

As in the previous sections, let Ω be an open subset of $\mathbb{R}^d$ with Lebesgue measure $M > 0$, $d\mu(x) := \dfrac{1}{M}dx$, where dx is the d-dimensional Lebesgue measure on Ω and let $K \subset \mathbb{R}^l$ be compact. Denote $P := \dfrac{1}{M}dx$.

We can now formulate the main theorem of the article.

Theorem 5. *Let $f\colon \mathbb{R}^d \supset \Omega \to K \subset \mathbb{R}^l$ be a Borel function with Young measure μ^f. Then μ^f is the probability distribution of the random variable $Y = f(U)$, where U has a uniform distribution on Ω.*

Proof. The distribution of a random variable Y is of the form: $\forall\, C \in \mathcal{B}(K)$, $P_{f(U)}(C) = P_U(f^{-1}(C))$. Let f be constant on Ω with value p and vanish on the complement of Ω. By theorem 4 we have $\mu^f = \delta_p$. For any $C \subseteq K$ we have

$$\mu^f(C) = \int_{\mathbb{R}^l} \mathbf{1}_C(p)\,\mathrm{d}\delta_p = \begin{cases} 1, & p \in C \\ 0, & p \notin C. \end{cases}$$

On the other hand,

$$P_U(f^{-1}(C)) = \int_{f^{-1}(C)} g_u \,\mathrm{d}P = \frac{1}{M} \int_{\{x: f(x) \in C\}} \mathrm{d}x = \begin{cases} \dfrac{1}{M} \cdot M = 1, & p \in C \\ \dfrac{1}{M} \cdot 0 = 0, & p \notin C. \end{cases}$$

Thus $P_Y = \mu^f$. This equality also holds when f is a simple function, due to the linearity of the integral. Since functions under consideration have values in the compact set K, Corollary 2 and the dominated convergence theorem yield the result for any Borel f. $\square$

4. Some applications and comments

Theorem 5 provides direct link between the Young measure basic concepts and the probability theory. This allows a wealth of probabilistic tools to be used to derive the explicit forms of the density functions of Young measures in many practically interesting cases. We illustrate this point with two examples.

Let Ω be as at the beginning of the previous section. Consider $\{\Omega\}$ – an open partition of Ω into at most countable number of open subsets $\Omega_1, \Omega_2, \ldots, \Omega_n, \ldots$ such that

(i) the elements of $\{\Omega\}$ are pairwise disjoint;

(ii) $\bigcup_i cl(\Omega_i) = cl(\Omega)$.

Let us consider functions $f_i\colon \Omega_i \to K \subset \mathbb{R}^d$, $i = 1, 2, \ldots$, with inverses f_i^{-1} that are continuously differentiable on $f(\Omega_i)$ and let $K_i := cl(f(\Omega_i))$ be compact. Denote for each $i = 1, 2, \ldots$ the Jacobian matrix of f_i^{-1} by $J_{f_i^{-1}}$.

Let a function $f\colon \Omega \to K$, with $K := (f(\Omega))$ compact, be such that

$$f(x) = \sum_i f_i(x)\mathbf{1}_{\Omega_i}(x)\ ,\ x \in \bigcup_i \Omega_i. \tag{2}$$

Then the following result holds.

Proposition 1. *The Young measure associated with Borel function f satisfying (2) is a homogeneous one and its density g with respect to the Lebesgue measure on K is of the following form*

$$g(y) = \frac{1}{M} \sum_{i:y\in K_i} |J_{f_i^{-1}}(y)| \tag{3}$$

The above result is a conclusion of our main Theorem 5 and the general probabilistic results concerning the distributions of the functions of random vectors/variables, (compare for instance [21] or [11]).

Observe, that the above Proposition 1 extends the results stated in Propositions 3.2, 3.4 in [17] which are proven there directly on the basis of Young measure notion.

Example 1. As an example of a more specific application of Proposition 1 let us consider the one-dimensional case ($d = 1$) and the following function $f\colon (0,1) \to (0,1)$:

$$f(x) = \sum_{i=2}^{\infty}(nx-1)\mathbf{1}_{\left[\frac{1}{n},\frac{1}{n-1}\right)}(x) \tag{4}$$

A graph of this function is presented in Fig. 1

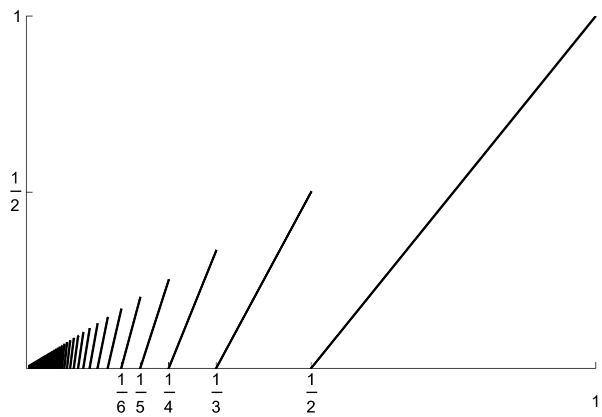

Figure 1: Graph of the Borel function f given by the formula (4)

The function given by (4) is of the form (2), thus by the Proposition 1, the Young measure associated with this f has the density function g that is a piecewise constant one and given by the following formula:

$$g(y) = (H_n - 1)\mathbf{1}_{\left[\frac{1}{n},\frac{1}{n-1}\right)}(y)\ ,\ y \in (0,1) \tag{5}$$

where H_n stands for the n-th harmonic number. The graph of this density function is presented in Fig. 2.

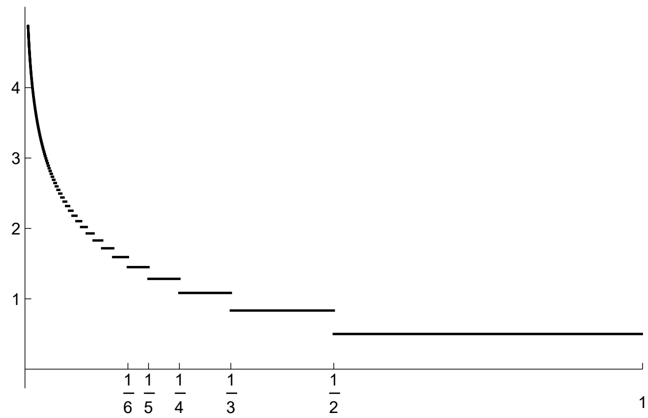

Figure 2: Graph of the density function g of the Young measure related to the function f given by (4). Function g is given by the formula (5)

Due to the main Theorem 5 various other results e.g. related to Borel functions with different dimensions of their domains and images, can be obtained with the help of other known probabilistic results concerning distributions of functions of random variables or vectors. Such results are of particular importance in the engineering practice, as they allow us the determination of specific values of Young's functionals, either directly or, in the more complex cases, by the Monte Carlo simulation.

Theorem 5 is a theoretical basis of the following computer routine enabling us to the Monte Carlo simulation of the Young measures associated with Borel functions. Considered function f is defined on a bounded interval $I = (a, b) \subset \mathbb{R}$ divided into n subintervals $I_1, \ldots, I_n$ of equal length. We first draw an interval and then choose at random an argument from this interval.

The routine is as follows.

SET $k = 1$;

SET JUMP $= \dfrac{b-a}{n}$

WHILE $k \leqslant N$ DO STEP 1 TO STEP 4

STEP 1. SET $k = \text{RANDOMI}(\{1, \ldots, n\})$
STEP 2. SET $t = a + (k - \text{RANDOM}([0, 1]) \cdot \text{JUMP}$
STEP 3. SET $z[k] = f(t)$
STEP 4. SET $k = k + 1$
SET SAMPLE=$(z[1], \ldots, z[N])$
RETURN SAMPLE

Detailed analysis of the problem for piecewise linear functions together with comparison of various routines is in [9], while in [10] one can find an application of this method in evaluation the values of the Young functional.

We illustrate the above idea in the following example.

Example 2. Consider a function of the form

$$f(x) = \begin{cases} 3x, & x \in \left(0, \frac{1}{6}\right], \\ \frac{3}{2}x + \frac{1}{4}, & x \in \left(\frac{1}{6}, \frac{1}{2}\right], \\ -\frac{3}{2}x + \frac{7}{4}, & x \in \left(\frac{1}{2}, \frac{5}{6}\right], \\ -3x + 3, & x \in \left(\frac{5}{6}, 1\right). \end{cases}$$

generating an oscillating sequence (f_n) with n-th element having n such 'teeth'. The Young measure generated by this sequence is absolutely continuous with respect to the Lebesgue measure on the range of f_n, $n \in \mathbb{N}$, and has density of the form

$$g(y) = \begin{cases} \frac{2}{3}, & y \in \left(0, \frac{1}{2}\right], \\ \frac{4}{3}, & y \in \left(\frac{1}{2}, 1\right], \end{cases}$$

see page 131 in [20].

We now apply our procedure and compare obtained experimental results with the theoretical ones. As an index of quality of the generated distributions we use Pearson goodness-of-fit χ^2-statistic.

We set the number of subintervals $n = 200$ and the sample size $N = 5000$. Further, we assume the significance level of the goodness-of-fit test as $\alpha = 0.1$. The critical value of the test in this situation is 224.957.

It turns out that with these initial conditions our routine produces a histogram which agrees very well with theoretical results. Namely, the value of the

χ^2-statistic is equal to 197.056. This means that our generator produces the experimental distribution that can be considered similar or even identical with the theoretical one. Moreover, the mean relative error is equal to 1.1%.

5. Conclusions

Young measures are objects discovered while investigating the existence of minima of integral functionals. Young measures are in fact measure-valued mappings and in general those of them which arise from physical or engineering problems are weak* limits of sequences of rapidly oscillating functions, divergent in the norm topology. This makes the task of obtaining an explicit form of such limits in specific cases difficult, if possible. The Theorem 5 provides quite general, probabilistic in nature, characterization of Young measures associated with bounded Borel functions. This allows using probabilistic tools in considering problems involving Young measures. In particular, it justifies the use of powerful Monte Carlo simulation in modeling the relevant phenomena.

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