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Nonconvex interval-valued multitime control problems with first-order PDE constraints via the minimax exact penalty method

Tadeusz ANTCZAK

The growing use of optimization models to help decision-making has created a demand for such tools that allow solving more real models of processes related to human activity in which hypotheses are not verified in a way specific for classical optimization. And just interval-valued optimization problems were developed for formulating such real-world problems which are usually not well defined and they contain uncertain data. In this paper, an interval-valued multitime control problem with first-order PDE constraints is considered. Then, the minimax exact penalty function method is used for solving the aforesaid interval-valued extremum problem. The most important property of all exact penalty function methods, that is, exactness of the penalization, is generalized to the case when one of such methods is used for solving an interval-valued multitime control problem. Namely, under appropriate invexity hypotheses, it is examined in the case of the minimax exact penalty function method which is used for solving the aforesaid interval-valued multitime control problem with the first-order PDE constraints.

Key words: interval-valued multitime control optimization problem with first-order PDE constraints, minimax exact penalty function method, exactness of the penalization, *LU*-optimal solution, invexity

1. Introduction

In many real-life problems connecting with human activity, the occurrence of vagueness and/or imprecision in the data is inevitable due to the increasing complexity of various areas in which modern humans live and work. Therefore, deterministic optimization models remain inadequate or insufficient to optimization problems that are models of such real-life problems. Hence, more important

T. Antczak (e-mail: tadeusz.antczak@wmii.uni.lodz.pl) is with Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland.

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optimization problems which grow rapidly are uncertain optimization problems, i.e. extremum problems with data uncertainty.

Among various techniques used for solving extremum problems, interval-valued optimization is a branch of mathematics which deals with uncertainty in optimization problems. In an interval-valued optimization problem, uncertainty may appear in the form of closed intervals either in the objective function or in at least one of the constraints or in both inequality and equality constraints. The interval optimization plays an important role in examining an optimization problem in the face of data uncertainty and allows to find its solution. Since in most real-world situations the model coefficients are not exactly known, the coefficients in the interval-valued extremum problems are taken as closed intervals. Although the specifications of closed intervals may still be judged as a subjective viewpoint, however, the modelling of real-life situations under interval uncertainty requires no additional information, such as membership function and probability distribution. In fact, the bounds of uncertain data (i.e., determining the closed intervals to bound the possible observed data) are, in general, easier to handle than specifying the Gaussian distributions in stochastic optimization problems. This is a consequence of the fact that they do not require the specification or the assumption of probabilistic distributions (as in stochastic programming) or possibilistic distributions (as in fuzzy programming). In the recent literature, optimality and duality results can be found where imprecision in real problems is formulated using interval analysis as a mathematical tool (see, for example, Antczak [5, 7], Debnath and Pokharna [10], Ghosh et al. [11], Guo et al. [12], Jayswal et al. [16], Rahman et al. [25], Treanţă [28], Wu [30, 31], and many others).

Besides this, one more important class of extremum problems is in the center of attention of modern researchers who are paying their attention to study the multitime control problems involving first-order PDE constraints. This is a consequence of the fact that aforesaid extremum problems arise in wide areas of research in pure and applied sciences and in the new technology as well. This is the reason why, in recent years, the multitime control problems involving first-order PDE constraints have attracted some attention in literature (see, for example, Jayswal and Preeti [14, 15], Mititelu [19], Mititelu and Treanţă [21, 22], Pitea and Postolache [23], Pitea et al. [24], Treanţă [26, 27], Udrişte et al. [29]). Jayswal and Preeti [14] formulated a penalized optimization problem with the l_1 exact penalty function method associated with the multitime control problem with first-order PDE constraints and showed that both the problems attained their optimality at the same solution under some suitable assumptions. Treanţă [26, 27] investigated single and multiobjective multitime control problems for which, under generalized convexity assumptions, he established optimality conditions and saddle point criteria.

One of the methods for solving constrained optimization problems are exact penalty functions methods. The exact penalty methods for finding optimal solutions of constrained optimization problems are based on the construction of a function which unconstrained minimizing points are also an optimal solution of the constrained extremum problem. One of frequently used types of exact penalty functions is the minimax exact penalty function method, which has been introduced to optimization theory by Bandler and Charalambous [9] The minimax exact penalty function method has been used for solving various classes of optimization problems (see, for example, Antczak [2–4, 6], Charalambous [8], Jayswal and Choudhury [13]). In recent years, the exact penalty functions methods have been used also for solving various classes of nonlinear interval-valued optimization problems and their most important property, that is, exactness of the penalization, has been analyzed also for such interval-valued extremum problems (see, for example, Antczak [5–7]).

The aim of this paper is to immunize against all the possible uncertainty concerned with the considered interval-valued multitime control problem with first-order PDE constraints and find its LU-solutions in an efficient way. In order to do this, we use the minimax exact penalty function method by the means of which we reduce the complexity of the aforesaid constrained interval-valued extremum problem and transform it into an equivalent unconstrained interval-valued penalized multitime control problem. Then, we analyze the most important property from a practical point of view of all exact penalty function methods, that is, exactness of the penalization. We define and extend the aforesaid property to the case when such a method is used for solving an interval-valued multitime control problem with first-order PDE constraints. Thus, under appropriate invexity hypotheses, we establish the equivalence between LU-solutions in the multitime control optimization problems with first-order PDE constraints and unconstrained interval-valued penalized multitime control problem which is constructed in the minimax exact penalty function method used in the paper. We illustrate the aforesaid result by an example of an invex interval-valued multitime control problem with first-order PDE constraints.

2. Preliminaries and interval-valued multitime control problem

The following convention for equalities and inequalities will be used in the paper.

For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;

(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \ldots, n$;

(iii) $x \geqslant y$ if and only if $x_i \geqslant y_i$ for all $i = 1, 2, \ldots, n$.

Now, we give some basic definitions and results from interval analysis on the lines Alefeld and Herzberger [1].

Let $I(\mathbb{R})$ be a class of all closed and bounded intervals in $\mathbb{R}$. Throughout this paper, when we say that V is a closed interval, we mean that V is also bounded in $\mathbb{R}$. If V is a closed interval, we use the notation $V = [v^L, v^U]$, where v^L and v^U mean the lower and upper bounds of V, respectively. In other words, if $V = [v^L, v^U] \in I(\mathbb{R})$, then $V = [v^L, v^U] = \left\{x \in \mathbb{R} : v^L \leqslant x \leqslant v^U\right\}$. If $v^L = v^U = v$, then $V = [v, v] = v$ is a real number.

Let $V = [v^L, v^U]$, $W = [w^L, w^U]$, then, by definition, we have:

i) $V + W = \{v + w : v \in V \text{ and } w \in W\} = [v^L + w^L, v^U + w^U]$,

ii $-V = \{-v : v \in V\} = [-v^U, -v^L]$,

iii) $V - W = V + (-W) = \{v - w : v \in V \text{ and } w \in W\} = [v^L - w^U, v^U - w^L]$,

iv) $k + V = \{k + v : v \in V\} = [k + v^L, k + v^U]$, where k is a real number,

v) $kV = \begin{cases} \left[kv^L, kv^U\right] & \text{if } k > 0, \\ \left[kv^U, kv^L\right] & \text{if } k \leqslant 0. \end{cases}$ where k is a real number.

In interval mathematics, an order relation is often used to rank interval numbers and it implies that an interval number is better than another but not that one is larger than another. For $V = [v^L, v^U]$ and $W = [w^L, w^U]$, we write $V \preceq_{LU} W$ if and only if $v^L \leqslant w^L$ and $v^U \leqslant w^U$. It is easy to see that $\preceq_{LU}$ (which was defined by Wu [30]) is a partial ordering on $I(\mathbb{R})$. Also, we can write $V \prec_{LU} W$ if and only if $V \preceq_{LU} W$ and $V \neq W$. Equivalently, $V \prec_{LU} W$ if and only if $\left(v^L < w^L,\ v^U \leqslant w^U\right)$ or $\left(v^L \leqslant w^L,\ v^U < w^U\right)$ or $\left(v^L < w^L,\ v^U < w^U\right)$.

Definition 1. *Let S be a nonempty subset of $\mathbb{R}^n$. A function $\varphi : S \to I(\mathbb{R})$ is called an interval-valued function. The interval-valued function φ can also be written as $\varphi(x) = \left[\varphi^L(x), \varphi^U(x)\right]$ with φ^L, $\varphi^U : S \to \mathbb{R}$ such that*

$$\varphi^L(x) \leqslant \varphi^U(x) \quad \textit{for each} \quad x \in S. \tag{1}$$

The real-valued functions φ^L and φ^U are called the lower endpoint function and the upper endpoint function, respectively.

Now, we give the following useful lemma in proving one of the main results in this paper, the simple proof of which is omitted in this paper.

Lemma 1. *(Antczak [2]) Let ψ_k, $k = 1, \ldots, r$, be real-valued functions defined on a nonempty set $S \subseteq \mathbb{R}^n$. For each $z \in S$, one has*

$$\max_{1 \leqslant k \leqslant r} \psi_k(z) = \max_{\theta \in \Delta} \sum_{k=1}^{r} \theta_k \psi_k(z),$$

where $\Delta := \left\{\theta = (\theta_1, \ldots, \theta_r) \in \mathbb{R}^r :\ \theta_k \geqslant 0,\ k = 1, \ldots, r,\ \sum_{k=1}^{r} \theta_k = 1\right\}$.

Before formulating the interval-valued multitime control problem (IVMCP) with first-order PDE constraints considered in the paper, we introduce some denotations used in the sequel.

- Let $(\mathcal{M}, g)$ be a given n-dimensional complete Riemannian manifold and $(\mathcal{T}, h)$ be a given a q-dimensional Riemannian manifold.
- Consider $t = \left(t^\beta\right)$, $\beta = 1, \ldots, q$, and $x = (x^\alpha)$, $\alpha = 1, \ldots, n$, the local coordinates on $(\mathcal{T}, h)$ and $(\mathcal{M}, g)$, respectively.
- Let the hyperparallelepiped Ω_{t_0,t_1} in $\mathcal{T}$ be fixed the diagonally opposite points $t_0 = \left(t_0^\beta\right)$ and $t_1 = \left(t_1^\beta\right)$, $\beta = 1, \ldots, q$, and $t = \left(t^\beta\right) \in \Omega_{t_0,t_1}$ be a multiple parameter of evolution of a multitime.
- $d\omega := \sqrt{\det h} dt^1 \wedge dt^2 \wedge \ldots \wedge dt^q$ represents the volume element on $\mathcal{T} \supset \Omega_{t_0,t_1}$ and, in order to simplify the writing for our control problem formulation, it was assumed $\mathcal{T} = R^q$, without loss the generality (hence, $\det h = 1$).
- Denote by $\mathcal{X}$ the space of piecewise smooth state functions $x\colon \Omega_{t_0,t_1} \subset \mathcal{T} \to \mathcal{M}$ endowed with the distance denote by $d\left(x, \overline{x}\right) = d\left(x(\cdot), \overline{x}(\cdot)\right) = \sup_{t\in\Omega} d_G\left(x(t), \overline{x}(t)\right)$, where $d_G\left(x(t), \overline{x}(t)\right)$ is a geodesic distance in (M, G), in this sense, $(\mathcal{X}, d)$ is a metric space.
- Let $\mathcal{U} \subset \mathbb{R}^s$ the space piecewise continuous control functions $u\colon \Omega_{t_0,t_1} \subset \mathcal{T} \to \mathcal{U}$ equipped with the uniform norm $\|\cdot\|_\infty$.

Throughout this paper, let us introduce the following denotations: $x = x(t)$, $u = u(t)$, $\pi_{xu}(t) = (t, x(t), u(t))$, $\pi_{\overline{xu}}(t) = (t, \overline{x}(t), \overline{u}(t))$, $\pi_{\widehat{xu}}(t) = (t, \widehat{x}(t), \widehat{u}(t))$, $\pi_{\widetilde{xu}}(t) = (t, \widetilde{x}(t), \widetilde{u}(t))$, $c = c(t)$, $A = \{1, \ldots, n\}$, $B = \{1, \ldots, q\}$.

In the paper, we consider the interval-valued multitime control problem (IVMCP) with first-order PDE constraints defined by:

$$
\begin{aligned}
&\min_{(x(\cdot),u(\cdot))} F\left(x(\cdot), u(\cdot)\right) = \int\limits_{\Omega_{t_0,t_1}} f\left(t, x(t), u(t)\right) \mathrm{d}\omega = \\
&\int\limits_{\Omega_{t_0,t_1}} \left[f^L\left(t, x(t), u(t)\right), f^U\left(t, x(t), u(t)\right)\right] \mathrm{d}\omega \\
&s.t.\ \ G_j\left(t, x(t), u(t)\right) \leqslant 0,\ \ j \in J = \{1, \ldots, m\}, \\
&\quad \frac{\partial x^\alpha}{\partial t^\beta} = H_\beta^\alpha\left(t, x(t), u(t)\right),\ \ \alpha \in A,\ \ \beta \in B, \\
&\qquad t \in \Omega_{t_0,t_1},\ x\left(t_0\right) = x_0,\ x\left(t_1\right) = x_1,
\end{aligned}
\tag{IVMCP}
$$

where $f\colon \Omega_{t_0,t_1} \times \mathcal{M} \times \mathcal{U} \to I(\mathbb{R})$ is a weakly continuously differentiable interval-valued functional with its lower and the upper (endpoint) functionals

f^L, $f^U\colon \Omega_{t_0,t_1} \times \mathcal{M} \times \mathcal{U} \to \mathbb{R}$, $G = \left(G_j\right) : \Omega_{t_0,t_1} \times \mathcal{M} \times \mathcal{U} \to \mathbb{R}^m$, $j \in J$, $H_\beta = \left(H_\beta^\alpha\right) : \Omega_{t_0,t_1} \times \mathcal{M} \times \mathcal{U} \to \mathbb{R}^n$, $\alpha \in A$, $\beta \in B$, are continuously differentiable functionals.

Let

$$\Gamma\left(\Omega_{t_0,t_1}\right) = \Big\{(x(t), u(t)) \in \mathcal{X} \times \mathcal{U} : t \in \Omega_{t_0,t_1},\ x\left(t^0\right) = x_0,\ x\left(t^1\right) = x_1,$$
$$G_j(t, x(t), u(t)) \leqslant 0, \quad j = 1, \ldots, m, \quad \frac{\partial x^\alpha}{\partial t^\beta} = H_\beta^\alpha\left(t, x(t), u(t)\right),$$
$$\alpha = 1, \ldots, n, \quad \beta = 1, \ldots, q\Big\}$$

denote the set all feasible solutions of the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints.

We define an optimal solution for the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints as an LU-optimal solution in the following sense:

Definition 2. *A feasible solution $(\overline{x}(\cdot), \overline{u}(\cdot))$ is said to be an LU-optimal solution in the considered interval-valued multitime control problem (IVMCP) if there does not exist other $(x(\cdot), u(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ such that the inequality*

$$\int_{\Omega_{t_0,t_1}} f\left(t, x(t), u(t)\right) \mathrm{d}\omega \prec_{LU} \int_{\Omega_{t_0,t_1}} f\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega \tag{2}$$

holds.

Remark 1. *If we use the definition of the relation $\prec_{LU}$, then LU-optimality of $(\overline{x}(t), \overline{u}(t))$ in (IVMPC) means that there does not exist other $(x(t), u(t)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ such that the following system of inequalities*

$$\left(\int_{\Omega_{t_0,t_1}} f^L\left(t, x(t), u(t)\right) \mathrm{d}\omega < \int_{\Omega_{t_0,t_1}} f^L\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega \wedge \right.$$
$$\left. \int_{\Omega_{t_0,t_1}} f^U\left(t, x(t), u(t)\right) \mathrm{d}\omega \leqslant \int_{\Omega_{t_0,t_1}} f^U\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega \right) \tag{3}$$

$$\textit{or}\ \left(\int_{\Omega_{t_0,t_1}} f^L(t,x(t),u(t))\,\mathrm{d}\omega \leqslant \int_{\Omega_{t_0,t_1}} f^L(t,\overline{x}(t),\overline{u}(t))\,\mathrm{d}\omega \wedge \int_{\Omega_{t_0,t_1}} f^U(t,x(t),u(t))\,\mathrm{d}\omega < \int_{\Omega_{t_0,t_1}} f^U(t,\overline{x}(t),\overline{u}(t))\,\mathrm{d}\omega\right)$$

$$\textit{or}\ \left(\int_{\Omega_{t_0,t_1}} f^L(t,x(t),u(t))\,\mathrm{d}\omega < \int_{\Omega_{t_0,t_1}} f^L(t,\overline{x}(t),\overline{u}(t))\,\mathrm{d}\omega \wedge \int_{\Omega_{t_0,t_1}} f^U(t,x(t),u(t))\,\mathrm{d}\omega < \int_{\Omega_{t_0,t_1}} f^U(t,\overline{x}(t),\overline{u}(t))\,\mathrm{d}\omega\right)$$

is satisfied. In other words, the relation (2) is equivalent to the aforesaid system (3) of inequalities.

Definition 3. *It is said that the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints is solvable if there exists at least one its LU-optimal solution.*

Now, we give the definition of invexity for a functional which was given by Mititelu [19]. Let $\Psi\colon \mathcal{X}\times\mathcal{U} \to \mathbb{R}$ be defined by $\Psi(x,u) = \int_{\Omega_{t_0t_1}} \varphi\left(t,x(t),x_\beta(t),u(t)\right)\mathrm{d}\omega$, where $x_\beta(t) = \dfrac{\partial x}{\partial t^\beta}$ is the derivative of $x(t)$ and the scalar function φ is of C^1-class. For notational convenience, we use $\varphi(t,x,u)$ for $\varphi\left(t,x(t),x_\beta(t),u(t)\right)$. Further, let $\eta := (\eta_1,\ldots,\eta_n) : \Omega_{t_0t_1}\times(\mathcal{X}\times\mathcal{U})^2 \to \mathbb{R}^n$ and $\xi := (\xi_1,\ldots,\xi_s) : \Omega_{t_0t_1}\times(\mathcal{X}\times\mathcal{U})^2 \to \mathbb{R}^s$. The following definition introduces the concept of invexity for the functional Ψ.

Definition 4. *A functional* $\Psi(x,u) = \int_{\Omega_{t_0t_1}} \varphi\left(t,x(t),x_\beta(t),u(t)\right)\mathrm{d}\omega$ *is said to be (strictly) invex at* $(\overline{x}(t),\overline{u}(t)) \in \mathcal{X}\times\mathcal{U}$ *on* $\mathcal{X}\times\mathcal{U}$ *with respect to functions* $\eta \in \mathbb{R}^n$ *of* C^1*-class, where* $\eta\big|_{\partial\Omega_{t_0,t_1}} = 0$ *(*$\partial\Omega_{t_0,t_1}$ *denotes the boundary of* Ω_{t_0,t_1}*),* *and* $\xi \in \mathbb{R}^s$ *of* C^0*-class with* $\xi\big|_{\partial\Omega_{t_0,t_1}} = 0$ *such that the following inequality*

$$\begin{aligned}
&\int_{\Omega_{t_0 t_1}} \varphi\left(t, x(t), x_\beta(t), u(t)\right) \mathrm{d}\omega - \int_{\Omega_{t_0 t_1}} \varphi\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \mathrm{d}\omega \\
&\quad \geqslant \int_{\Omega_{t_0 t_1}} \Bigg(\eta\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right) \frac{\partial \varphi}{\partial x}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \\
&\qquad + D_\beta\left(\eta\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right)\right) \frac{\partial \varphi}{\partial x_\beta}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \\
&\qquad + \xi\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right) \frac{\partial \varphi}{\partial u}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right)\Bigg) \mathrm{d}\omega \ (>)
\end{aligned}$$

holds for all $(x(t), u(t)) \in \mathcal{X} \times \mathcal{U}$*,* $((x(t), u(t)) \neq (\overline{x}(t), \overline{u}(t)))$*. If the inequality above is satisfied for every* $(\overline{x}(t), \overline{u}(t)) \in \mathcal{X} \times \mathcal{U}$*,* $((x(t), u(t)) \neq (\overline{x}(t), \overline{u}(t)))$*, then* Ψ *is said to be (strictly) invex on* $\mathcal{X} \times \mathcal{U}$ *with respect to functions* η *and* ξ*.*

Now, we generalize the aforesaid definition of invexity to the case of an interval-valued functional.

Definition 5. *An interval-valued functional* $\int_{\Omega_{t_0,t_1}} f(t, \cdot, \cdot, \cdot)\mathrm{d}\omega$ *is said to be invex at* $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \mathcal{X} \times \mathcal{U}$ *on* $\mathcal{X} \times \mathcal{U}$ *(with respect to* η *and* ξ*) if its lower and the upper (endpoint) functionals* f^L*,* f^U *are (strictly) invex at* $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \mathcal{X} \times \mathcal{U}$ *on* $\mathcal{X} \times \mathcal{U}$ *with respect to the same* η *and* ξ*. In other words,* $\int_{\Omega_{t_0,t_1}} f(t, \cdot, \cdot, \cdot)\mathrm{d}\omega$ *is said to be an invex interval-valued functional at* $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \mathcal{X} \times \mathcal{U}$ *on* $\mathcal{X} \times \mathcal{U}$ *if the inequalities*

$$\begin{aligned}
&\int_{\Omega_{t_0 t_1}} f^L\left(t, x(t), x_\beta(t), u(t)\right) \mathrm{d}\omega - \int_{\Omega_{t_0 t_1}} f^L\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \mathrm{d}\omega \\
&\quad \geqslant \int_{\Omega_{t_0 t_1}} \Bigg(\eta\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right) \frac{\partial f^L}{\partial x}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \\
&\qquad + D_\beta\left(\eta\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right)\right) \frac{\partial f^L}{\partial x_\beta}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \\
&\qquad + \xi\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right) \frac{\partial f^L}{\partial u}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right)\Bigg) \mathrm{d}\omega, \ (>)
\end{aligned} \tag{4}$$

$$\int_{\Omega_{t_0 t_1}} f^U\left(t, x(t), x_\beta(t), u(t)\right) \mathrm{d}\omega - \int_{\Omega_{t_0 t_1}} f^U\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0 t_1}} \Bigg(\eta\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right) \frac{\partial f^U}{\partial x}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right)$$
$$+ D_\beta\left(\eta\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right)\right) \frac{\partial f^U}{\partial x_\beta}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right)$$
$$+ \xi\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right) \frac{\partial f^U}{\partial u}\left(t, \overline{x}(t), \overline{x}_\beta(t), \overline{u}(t)\right) \Bigg) \mathrm{d}\omega \quad (>) \tag{5}$$

hold for all $\left(x(\cdot), u(\cdot)\right) \in \mathcal{X} \times \mathcal{U}$, $\left(\left(x(\cdot), u(\cdot)\right) \neq \left(\overline{x}(\cdot), \overline{u}(\cdot)\right)\right)$.

Now, we give the necessary optimality conditions of the Karush-Kuhn-Tucker type for the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints (see, for example, Mititelu and Treanţă [21], Treanţă [28]).

Theorem 1. *(Necessary Optimality Conditions): Let* $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ *be a normal LU-solution in the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints. Then there exist* $\overline{\lambda}^L \in \mathbb{R}$, $\overline{\lambda}^U \in \mathbb{R}$ *and piecewise smooth functions* $\overline{\mu}\colon \Omega_{t_0,t_1} \to \mathbb{R}^m_+$, $\overline{\vartheta}\colon \Omega_{t_0,t_1} \to \mathbb{R}^{qn}$ *such that the following conditions (with summation over the repeated indices)*

$$\overline{\lambda}^L \frac{\partial f^L}{\partial x^\alpha}\left(\pi_{\overline{xu}}(t)\right) + \overline{\lambda}^U \frac{\partial f^U}{\partial x^\alpha}\left(\pi_{\overline{xu}}(t)\right) + \overline{\mu}_j(t) \frac{\partial G_j}{\partial x^\alpha}\left(\pi_{\overline{xu}}(t)\right)$$
$$+ \overline{\vartheta}^\beta_\alpha \frac{\partial H^\alpha_\beta}{\partial x^\alpha}\left(\pi_{\overline{xu}}(t)\right) + \frac{\partial \overline{\vartheta}^\beta_\alpha}{\partial t^\beta} = 0, \quad \alpha = 1, \ldots, n, \;\; \beta = 1, \ldots, q, \tag{6}$$

$$\overline{\lambda}^L \frac{\partial f^L}{\partial u^r}\left(\pi_{\overline{xu}}(t)\right) + \overline{\lambda}^U \frac{\partial f^U}{\partial u^r}\left(\pi_{\overline{xu}}(t)\right) + \overline{\mu}_j(t) \frac{\partial G_j}{\partial u^r}\left(\pi_{\overline{xu}}(t)\right)$$
$$+ \overline{\vartheta}^\beta_\alpha \frac{\partial H^\alpha_\beta}{\partial u^r}\left(\pi_{\overline{xu}}(t)\right) = 0, \quad \alpha = 1, \ldots, n, \;\; r = 1, \ldots, s, \tag{7}$$

$$\overline{\mu}_j(t) G_j\left(\pi_{\overline{xu}}(t)\right) = 0, \quad t \in \Omega_{t_0,t_1}, \;\; j = 1, \ldots, m, \tag{8}$$

$$\overline{\lambda}^L > 0, \quad \overline{\lambda}^U > 0, \quad \overline{\mu}_j(t) \geqslant 0, \quad j = 1, \ldots, m \tag{9}$$

are fulfilled for all $t \in \Omega_{t_0,t_1}$, *except at discontinuities.*

Definition 6. *If* $\overline{\lambda}^L > 0$ *and* $\overline{\lambda}^U > 0$, *then* $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$, *satisfying the conditions (6)–(9), is called a normal LU-optimal solution of (IVMCP).*

Definition 7. $\left((\overline{x}(\cdot),\overline{u}(\cdot)),\overline{\lambda}^{L},\overline{\lambda}^{U},\overline{\mu}(\cdot),\overline{\xi}(\cdot)\right)\in\Gamma\left(\Omega_{t_0,t_1}\right)\times\mathbb{R}_{+}^{2}\times\mathbb{R}_{+}^{m}\times\mathbb{R}^{qn}$ *is said a Karush-Kuhn-Tucker point (a KKT point, for short) of the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints if the necessary optimality conditions (6)–(9) are fulfilled at* $(\overline{x}(\cdot),\overline{u}(\cdot))$ *with scalars* $\overline{\lambda}^{L}\in\mathbb{R}$, $\overline{\lambda}^{U}\in\mathbb{R}$ *and piecewise smooth functions* $\overline{\mu}\colon\Omega_{t_0,t_1}\to\mathbb{R}_{+}^{m}$, $\overline{\xi}\colon\Omega_{t_0,t_1}\to\mathbb{R}^{qn}$.

3. Minimax exact penalty function method

One of the methods which are used in the literature for finding optimal solutions of constrained extremum problems are exact penalty function methods. These methods are based on the construction of a so-called penalty function whose unconstrained minimizing points are also optimal solutions of the constrained optimization problem for all sufficiently large values of the penalty parameter. Hence, such methods of solving mathematical programming problems transform an original constrained extremum problem into a single unconstrained optimization problem.

Usually, these methods have been used in the literature for solving classical optimization problems which are deterministic models of real-world O.R. problems. However, in this paper, we show that one of the aforementioned methods is applicable for solving interval-valued multitime control optimization problems with first-order PDE constraints which are models of aforesaid control problems with uncertainty.

Therefore, if we use any exact penalty function method for solving the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints, then we have to construct a unconstrained interval-valued penalized multitime variational problem (IVP(c)) which is defined by:

$$\min P(x(\cdot),u(\cdot),c(\cdot))=F(x(\cdot),u(\cdot))+c(\cdot)p(x(\cdot),u(\cdot)),\quad (\text{IVP}(c))\qquad(10)$$

where $F\colon\mathbb{R}^{n}\to I(\mathbb{R})$ is an interval-valued objective function of the original interval-valued multitime control problem (IVMCP) with first-order PDE constraints, p is a suitable penalty function, $c(\cdot)$ is a penalty parameter. As it follows from the definition of (IVP(c)), the original interval-valued multitime control problem (IVMCP) is replaced by an unconstrained multitime variational problem (IVP(c)), whose the interval-valued objective function is the sum of a certain fuzzy “merit” function (which reflects the interval-valued objective function of the given interval-valued optimization problem) and the penalty term which reflects the constraint set which is defined, among others, by means of first-order PDE constraints. The interval-valued merit function is chosen as the original interval-valued objective function, while the penalty term is obtained by multiplying a suitable function, which represents the constraints (among others, the aforementioned first-order PDE constraints), by a positive parameter c, called the penalty parameter.

Since we use an exact penalty function method for solving the given interval-valued multitime control problem (IVMCP) with first-order PDE constraints, therefore, the penalty objective function of the unconstrained interval-valued multitime variational problem (IVP(c)) constructed in this method is an interval-valued function. Thus, it follows from (10) that we associate with the original interval-valued mathematical programming problem (IVMPC) the following unconstrained interval-valued penalized optimization problem defined as follows:

$$\min P(x(\cdot), u(\cdot), c) = \left[P^L(x(\cdot), u(\cdot), c), P^U(x(\cdot), u(\cdot), c)\right] = \qquad \text{(IVP}(c))$$

$$\left[F^L(x(\cdot), u(\cdot)) + cp(x(\cdot), u(\cdot)),\;\; F^U(x(\cdot), u(\cdot)) + cp(x(\cdot), u(\cdot))\right]. \tag{11}$$

Now, we define the property of exactness of the penalization for the interval-valued exact penalty function method used for solving the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints. Hence, in a natural way, we extend the definition of exactness of the penalization defined for the classical exact penalty function method in the deterministic case to the aforesaid interval-valued case.

Definition 8. *If a threshold value $\overline{c}(\cdot) > 0$ exists such that, for every $c(\cdot) > \overline{c}(\cdot)$,*

$$\arg\ LU\text{-}optimal \left\{F(x,u)\colon\ (x(\cdot), u(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)\right\} =$$

$$\arg\ LU\text{-}optimal \left\{P(x(\cdot), u(\cdot), c(\cdot))\colon\ (x(\cdot), u(\cdot)) \in \mathcal{X} \times \mathcal{U}\right\},$$

then $P(x(\cdot), u(\cdot), c(\cdot))$ is termed an interval-valued exact penalty function and, therefore, we call (IVP(c)), defined by (11), the penalized multitime variational problem with an interval-valued exact penalty function.

It is clear that, conceptually, if $P(\cdot, c)$ is an interval-valued exact penalty function, that is, the objective function in the interval-valued penalized multitime variational problem, we can find “constrained” LU-optimal solutions in the original interval-valued multitime control problem (IVMCP) with first-order PDE constraints by finding “unconstrained” LU-optimal solutions of the penalty function $P(\cdot, c)$, for sufficiently large values of the penalty parameter c. Thus, in this paper, one of the main purposes is to show that there is the threshold of the penalty parameter such that, for every penalty parameter exceeding this threshold, the aforesaid equivalence holds.

Before we introduce an interval-valued minimax exact penalty function method, we define the function G_j^+, $j \in J$, for every inequality constraint G_j, $j \in J$, as follows:

$$G_j^+(t, x(t), u(t)) = \begin{cases} 0 & \text{if } G_j(t, x(t), u(t)) \leqslant 0, \\ G_j(t, x(t), u(t)) & \text{if } G_j(t, x(t), u(t)) > 0. \end{cases} \tag{12}$$

As it follows from (12), any function G_j^+, $j \in J$, has a positive value whenever the constraint G_j is violated. Therefore, note that the function G_j^+ defined above has the penalty features relative to the single inequality constraint G_j since large violations in the inequality constraint G_j result in large values for the function G_j^+.

One of the most popular nondifferentiable exact penalty function methods for solving nonlinear mathematical programming problems is the minimax exact penalty function method. Now, we use the aforesaid exact penalty function method for solving the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints. Then, the definition of the unconstrained interval-valued multitime control problem constructed in the minimax exact penalty function method for the original interval-valued multitime control problem (IVMCP) with first-order PDE constraints is given as follows:

$$P_\infty((x(\cdot), u(\cdot)), c(\cdot)) = \left[\int_{\Omega_{t_0,t_1}} \Big(f^L(t, x(t), u(t)) + c(t) p(t, x(t), u(t)) \Big) \, \mathrm{d}\omega, \right.$$
$$\left. \int_{\Omega_{t_0,t_1}} \Big(f^U(t, x(t), u(t)) + c(t) p(t, x(t), u(t)) \Big) \, \mathrm{d}\omega \right] \to \min, \qquad (\text{IVMCP}_\infty(c))$$

where

$$p(t, x(\cdot), u(\cdot)) = \max \Big\{ G_j^+(t, x(\cdot), u(\cdot)),\ j \in J, \left| H_\beta^\alpha(t, x(\cdot), u(\cdot)) - \frac{\partial x^\alpha}{\partial t^\beta} \right|,$$
$$\alpha = 1, \ldots, n,\ \beta = 1, \ldots, q, \Big\}.$$

Now, we investigate the exactness property of the penalization for the minimax exact penalty function method in the case when this method is used for solving the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints. In other words, we show that any LU-optimal solution in the original interval-valued extremum problem (IVMCP) is also an LU-optimal solution in its associated interval-valued penalized optimization problem with minimax exact interval-valued penalty function constructed in the aforesaid method.

First, under appropriate invexity hypotheses, we prove that each KKT-point in the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints is also an LU-optimal solution in its interval-valued penalized optimization problem with the interval-valued minimax exact penalty function for sufficiently large penalty parameters exceeding the given threshold.

Theorem 2. *Let $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ be a Karush-Kuhn-Tucker point in the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints and the necessary optimality conditions (6)–(9) be fulfilled with Lagrange multipliers $\overline{\lambda}^L \in \mathbb{R}_+$, $\overline{\lambda}^U \in \mathbb{R}_+$, and $\overline{\mu}(\cdot) = \left(\overline{\mu}_j(\cdot)\right) : \Omega_{t_0,t_1} \to \mathbb{R}^m_+$, $\overline{\vartheta}_\beta(\cdot) = \left(\overline{\vartheta}^\beta_\alpha(\cdot)\right) : \Omega_{t_0,t_1} \to \mathbb{R}^n$, which are piecewise smooth functions, except at discontinuities. Further, assume that:*

a) $\int\limits_{\Omega_{t_0,t_1}} f(t,\cdot,\cdot)\mathrm{d}\omega$ is an invex interval-valued functional at $(\overline{x}(\cdot), \overline{u}(\cdot))$ on $\mathcal{X} \times \mathcal{U}$ with respect η and ξ,

b) $\int\limits_{\Omega_{t_0,t_1}} \left\langle \overline{\mu}_j(\cdot), G_j(t,\cdot,\cdot)\right\rangle \mathrm{d}\omega$, $j = 1,\dots,m$, are invex functionals at $(\overline{x}(\cdot), \overline{u}(\cdot))$ on $\mathcal{X} \times \mathcal{U}$ with respect η and ξ,

c) $\int\limits_{\Omega_{t_0,t_1}} \left\langle \overline{\vartheta}^\beta_\alpha(\cdot), H^\alpha_\beta(t,\cdot,\cdot) - \dfrac{\partial \overline{x}^\alpha}{\partial t^\beta}\right\rangle \mathrm{d}\omega$, $\beta = 1,\dots,q$, $\alpha = 1,\dots,n$, are invex functionals at $(\overline{x}(\cdot), \overline{u}(\cdot))$ on $\mathcal{X} \times \mathcal{U}$ with respect η and ξ.

If we assume that the penalty parameter $c(\cdot)$ is large enough, i.e. it satisfies the following condition

$$c(\cdot) \geqslant \sum_{j=1}^{m} \overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q} \left|\overline{\vartheta}^\beta_\alpha(\cdot)\right|, \tag{13}$$

then $(\overline{x}(\cdot), \overline{u}(\cdot))$ is a LU-minimizer in the interval-valued penalized control problem (IVMCP$_\infty(c)$) with the interval-valued minimax exact penalty function.

Proof. We proceed by contradiction. Suppose, contrary to the result, that $(\overline{x}(\cdot), \overline{u}(\cdot))$ is not a LU-minimizer in (IVMCP$_\infty(c)$). Then, by Definition 2, there exists $(\widetilde{x}(\cdot), \widetilde{u}(\cdot)) \in \mathcal{X} \times \mathcal{U}$ such that

$$P_\infty\big((\widetilde{x}(\cdot), \widetilde{u}(\cdot)), c(\cdot)\big) \prec_{LU} P_\infty\big((\overline{x}(\cdot), \overline{u}(\cdot)), c(\cdot)\big).$$

Thus, by the definition of the relation $\prec_{LU}$, the aforesaid inequality yields

$$\left(\int_{\Omega_{t_0,t_1}} P_\infty^L\big((\overline{x}(t),\overline{u}(t)),c(t)\big)\mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^L\big((\widetilde{x}(t),\widetilde{u}(t)),c(t)\big)\mathrm{d}\omega \wedge \right.$$

$$\left. \int_{\Omega_{t_0,t_1}} P_\infty^U\big((\overline{x}(t),\overline{u}(t)),c(t)\big)\mathrm{d}\omega \geqslant \int_{\Omega_{t_0,t_1}} P_\infty^U\big((\widetilde{x}(t),\widetilde{u}(t)),c(t)\big)\mathrm{d}\omega\right)$$

$$\text{or } \left(\int_{\Omega_{t_0,t_1}} P_\infty^L\big((\overline{x}(t),\overline{u}(t)),c(t)\big)\mathrm{d}\omega \geqslant \int_{\Omega_{t_0,t_1}} P_\infty^L\big((\widetilde{x}(t),\widetilde{u}(t)),c(t)\big)\mathrm{d}\omega \wedge \right.$$

$$\left. \int_{\Omega_{t_0,t_1}} P_\infty^U\big((\overline{x}(t),\overline{u}(t)),c(t)\big)\mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^U\big((\widetilde{x}(t),\widetilde{u}(t)),c(t)\big)\mathrm{d}\omega\right)$$

$$\text{or } \left(\int_{\Omega_{t_0,t_1}} P_\infty^L\big((\overline{x}(t),\overline{u}(t)),c(t)\big)\mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^L\big((\widetilde{x}(t),\widetilde{u}(t)),c(t)\big)\mathrm{d}\omega \wedge \right.$$

$$\left. \int_{\Omega_{t_0,t_1}} P_\infty^U\big((\overline{x}(t),\overline{u}(t)),c(t)\big)\mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^U\big((\widetilde{x}(t),\widetilde{u}(t)),c(t)\big)\mathrm{d}\omega\right).$$

Hence, by the definition of the minimax penalty functional P_∞, one has

$$\left(\int_{\Omega_{t_0,t_1}} \left(f^L\left(\pi_{\overline{x}\overline{u}}\right) + c(t)\max\left\{G_j^+\left(\pi_{\overline{x}\overline{u}}\right),\ j\in J,\ \left|H_\beta^\alpha\left(\pi_{\overline{x}\overline{u}}\right) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|,\right.\right.\right.$$

$$\left.\left.\alpha\in A,\ \beta\in B\right\}\right)\mathrm{d}\omega$$

$$> \int_{\Omega_{t_0,t_1}} \left(f^L\left(\pi_{\widetilde{x}\widetilde{u}}\right) + c(t)\max\left\{G_j^+\left(\pi_{\widetilde{x}\widetilde{u}}\right),\ j\in J,\ \left|H_\beta^\alpha\left(\pi_{\widetilde{x}\widetilde{u}}\right) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta}\right|,\right.\right.$$

$$\left.\left.\alpha\in A,\ \beta\in B\right\}\right)\mathrm{d}\omega \wedge$$

$$\int_{\Omega_{t_0,t_1}} \left(f^U(\pi_{\overline{x}\overline{u}}) + c(t) \max\left\{ G_j^+(\pi_{\overline{x}\overline{u}}),\ j \in J,\ \left| H_\beta^\alpha(\pi_{\overline{x}\overline{u}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta} \right|, \right.\right.$$

$$\left.\left. \alpha \in A,\ \beta \in B \right\} \right) \mathrm{d}\omega$$

$$\geqslant \int_{\Omega_{t_0,t_1}} \left(f^U(\pi_{\widetilde{x}\widetilde{u}}) + c(t) \max\left\{ G_j^+(\pi_{\widetilde{x}\widetilde{u}}),\ j \in J,\ \left| H_\beta^\alpha(\pi_{\widetilde{x}\widetilde{u}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \right|, \right.\right.$$

$$\left.\left.\left. \alpha \in A,\ \beta \in B \right\} \right) \mathrm{d}\omega \right)$$

$$\text{or} \left(\int_{\Omega_{t_0,t_1}} \left(f^L(\pi_{\overline{x}\overline{u}}) + c(t) \max\left\{ G_j^+(\pi_{\overline{x}\overline{u}}),\ j \in J,\ \left| H_\beta^\alpha(\pi_{\overline{x}\overline{u}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta} \right|, \right.\right.\right.$$

$$\left.\left. \alpha \in A,\ \beta \in B \right\} \right) \mathrm{d}\omega$$

$$\geqslant \int_{\Omega_{t_0,t_1}} \left(f^L(\pi_{\widetilde{x}\widetilde{u}}) + c(t) \max\left\{ G_j^+(\pi_{\widetilde{x}\widetilde{u}}),\ j \in J,\ \left| H_\beta^\alpha(\pi_{\widetilde{x}\widetilde{u}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \right|, \right.\right.$$

$$\left.\left. \alpha \in A,\ \beta \in B \right\} \right) \mathrm{d}\omega \ \wedge$$

$$\int_{\Omega_{t_0,t_1}} \left(f^U(\pi_{\overline{x}\overline{u}}) + c(t) \max\left\{ G_j^+(\pi_{\overline{x}\overline{u}}),\ j \in J,\ \left| H_\beta^\alpha(\pi_{\overline{x}\overline{u}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta} \right|, \right.\right.$$

$$\left.\left. \alpha \in A,\ \beta \in B \right\} \right) \mathrm{d}\omega$$

$$> \int_{\Omega_{t_0,t_1}} \left(f^U(\pi_{\widetilde{x}\widetilde{u}}) + c(t) \max\left\{ G_j^+(\pi_{\widetilde{x}\widetilde{u}}),\ j \in J,\ \left| H_\beta^\alpha(\pi_{\widetilde{x}\widetilde{u}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \right|, \right.\right.$$

$$\left.\left.\left. \alpha \in A,\ \beta \in B \right\} \right) \mathrm{d}\omega \right)$$

$$\text{or}\ \left(\int\limits_{\Omega_{t_0,t_1}} \left(f^L\left(\pi_{\overline{xu}}\right) + c(t) \max\left\{G_j^+\left(\pi_{\overline{xu}}\right),\ j \in J,\ \left|H_\beta^\alpha\left(\pi_{\overline{xu}}\right) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|,\right.\right.\right.$$

$$\left.\left.\alpha \in A,\ \beta \in B\right\}\right) \mathrm{d}\omega$$

$$> \int\limits_{\Omega_{t_0,t_1}} \left(f^L\left(\pi_{\widetilde{xu}}\right) + c(t) \max\left\{G_j^+\left(\pi_{\widetilde{xu}}\right),\ j \in J,\ \left|H_\beta^\alpha\left(\pi_{\widetilde{xu}}\right) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta}\right|,\right.\right.$$

$$\left.\left.\alpha \in A,\ \beta \in B\right\}\right) \mathrm{d}\omega \wedge$$

$$\int\limits_{\Omega_{t_0,t_1}} \left(f^U\left(\pi_{\overline{xu}}\right) + c \max\left\{G_j^+\left(\pi_{\overline{xu}}\right),\ j \in J,\ \left|H_\beta^\alpha\left(\pi_{\overline{xu}}\right) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|,\right.\right.$$

$$\left.\left.\alpha \in A,\ \beta \in B\right\}\right) \mathrm{d}\omega$$

$$> \int\limits_{\Omega_{t_0,t_1}} \left(f^U\left(\pi_{\widetilde{xu}}\right) + c(t) \max\left\{G_j^+\left(\pi_{\widetilde{xu}}\right),\ j \in J,\ \left|H_\beta^\alpha\left(\pi_{\widetilde{xu}}\right) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta}\right|,\right.\right.$$

$$\left.\left.\left.\alpha \in A,\ \beta \in B\right\}\right) \mathrm{d}\omega\right).$$

Thus, using $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ together with (12), we obtain

$$\left(\int\limits_{\Omega_{t_0,t_1}} f^L\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega > \int\limits_{\Omega_{t_0,t_1}} \left(f^L\left(\pi_{\widetilde{xu}}\right)\right.\right.$$

$$\left. + c(t) \max\left\{G_j^+\left(\pi_{\widetilde{xu}}\right),\ j \in J,\ \left|H_\beta^\alpha\left(\pi_{\widetilde{xu}}\right) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta}\right|,\ \alpha \in A,\ \beta \in B\right\}\right) \mathrm{d}\omega \wedge$$

$$\int\limits_{\Omega_{t_0,t_1}} f^U\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \geqslant \int\limits_{\Omega_{t_0,t_1}} \left(f^U\left(\pi_{\widetilde{xu}}\right)\right.$$

$$\left.\left. + c(t) \max\left\{G_j^+\left(\pi_{\widetilde{xu}}\right),\ j \in J,\ \left|H_\beta^\alpha\left(\pi_{\widetilde{xu}}\right) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta}\right|,\ \alpha \in A,\ \beta \in B\right\}\right) \mathrm{d}\omega\right)$$

$$\text{or} \left(\int_{\Omega_{t_0,t_1}} f^L(\pi_{\overline{xu}})\, \mathrm{d}\omega \geqslant \int_{\Omega_{t_0,t_1}} \Big(f^L(\pi_{\widetilde{xu}}) + c(t) \max\Big\{ G_j^+(\pi_{\widetilde{xu}}),\ j \in J,\ \Big| H_\beta^\alpha(\pi_{\widetilde{xu}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \Big|,\ \alpha \in A,\ \beta \in B \Big\} \Big) d\omega \wedge \right.$$

$$\left. \int_{\Omega_{t_0,t_1}} f^U(\pi_{\overline{xu}})\, \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} \Big(f^U(\pi_{\widetilde{xu}}) + c(t) \max\Big\{ G_j^+(\pi_{\widetilde{xu}}),\ j \in J,\ \Big| H_\beta^\alpha(\pi_{\widetilde{xu}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \Big|,\ \alpha \in A,\ \beta \in B \Big\} \Big) \mathrm{d}\omega \right)$$

$$\text{or} \left(\int_{\Omega_{t_0,t_1}} f^L(\pi_{\overline{xu}})\, \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} \Big(f^L(\pi_{\widetilde{xu}}) + c(t) \max\Big\{ G_j^+(\pi_{\widetilde{xu}}),\ j \in J,\ \Big| H_\beta^\alpha(\pi_{\widetilde{xu}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \Big|,\ \alpha \in A,\ \beta \in B \Big\} \Big) d\omega \wedge \right.$$

$$\left. \int_{\Omega_{t_0,t_1}} f^U(\pi_{\overline{xu}})\, \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} \Big(f^U(\pi_{\widetilde{xu}}) + c(t) \max\Big\{ G_j^+(\pi_{\widetilde{xu}}),\ j \in J,\ \Big| H_\beta^\alpha(\pi_{\widetilde{xu}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \Big|,\ \alpha \in A,\ \beta \in B \Big\} \Big) \mathrm{d}\omega \right).$$

Multiplying the above inequalities by the corresponding Lagrange multipliers $\overline{\lambda}^L > 0, \overline{\lambda}^U > 0$ associated to the endpoint functionals f^L and f^U of the interval-valued objective function and then adding the resulting inequalities, we get

$$\left(\int_{\Omega_{t_0,t_1}} \Big(\overline{\lambda}^L f^L(\pi_{\overline{xu}}) + \overline{\lambda}^U f^U(\pi_{\overline{xu}}) \Big) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} \Big(\overline{\lambda}^L f^L(\pi_{\widetilde{xu}}) + \overline{\lambda}^U f^U(\pi_{\widetilde{xu}}) \right.$$

$$\left. + \Big(\overline{\lambda}^L + \overline{\lambda}^U \Big) c(t) \max \Big\{ G_j^+(\pi_{\widetilde{xu}}), j \in J, \Big| H_\beta^\alpha(\pi_{\widetilde{xu}}) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \Big|, \alpha \in A, \beta \in B \Big\} \Big) \mathrm{d}\omega \right).$$

Since $\overline{\lambda}^L + \overline{\lambda}^U = 1$, the inequality above implies

$$\left(\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(\pi_{\overline{x}\overline{u}}\right) + \overline{\lambda}^U f^U\left(\pi_{\overline{x}\overline{u}}\right) \right) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(\pi_{\widetilde{x}\widetilde{u}}\right) + \overline{\lambda}^U f^U\left(\pi_{\widetilde{x}\widetilde{u}}\right) \right.\right.$$
$$\left.\left. + \, c(t) \max\left\{ G_j^+\left(\pi_{\widetilde{x}\widetilde{u}}\right),\ j \in J,\ \left| H_\beta^\alpha\left(\pi_{\widetilde{x}\widetilde{u}}\right) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \right|,\ \alpha \in A,\ \beta \in B \right\} \right) \mathrm{d}\omega \right). \tag{14}$$

By assumption, $\int_{\Omega_{t_0,t_1}} f(t,\cdot,\cdot)\mathrm{d}\omega$ is an invex interval-valued functional at $(\overline{x}(\cdot),\overline{u}(\cdot))$ on $\mathcal{X} \times \mathcal{U}$. Then, by Definition 5, the inequalities

$$\begin{aligned} \int_{\Omega_{t_0,t_1}} & \left(f^L\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - f^L\left(t,\overline{x}(t),\overline{u}(t)\right) \right) \mathrm{d}\omega \\ & \geqslant \int_{\Omega_{t_0,t_1}} \eta\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \frac{\partial f^L}{\partial x^\alpha}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega \\ & \quad + \int_{\Omega_{t_0,t_1}} \xi\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \frac{\partial f^L}{\partial u^r}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega, \end{aligned} \tag{15}$$

$$\begin{aligned} \int_{\Omega_{t_0,t_1}} & \left(f^U\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - f^U\left(t,\overline{x}(t),\overline{u}(t)\right) \right) \mathrm{d}\omega \\ & \geqslant \int_{\Omega_{t_0,t_1}} \eta\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \frac{\partial f^U}{\partial x^\alpha}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega \\ & \quad + \int_{\Omega_{t_0,t_1}} \xi\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \frac{\partial f^U}{\partial u^r}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega \end{aligned} \tag{16}$$

hold. Multiplying (15) and (16) by the corresponding Lagrange multipliers $\overline{\lambda}^L > 0$, $\overline{\lambda}^U > 0$ and adding both sides of the resulting inequalities, we get

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \overline{\lambda}^L f^L\left(t,\overline{x}(t),\overline{u}(t)\right)\right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^U f^U\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \overline{\lambda}^U f^U\left(t,\overline{x}(t),\overline{u}(t)\right)\right) \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \eta\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \overline{\lambda}^L \frac{\partial f^L}{\partial x^\alpha}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \xi\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \overline{\lambda}^L \frac{\partial f^L}{\partial u^r}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \eta\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \overline{\lambda}^U \frac{\partial f^U}{\partial x^\alpha}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \xi\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \overline{\lambda}^U \frac{\partial f^U}{\partial u^r}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega. \tag{17}$$

Further, $\int_{\Omega_{t_0,t_1}} \left\langle \overline{\mu}_j(\cdot), G_j(t,\cdot,\cdot)\right\rangle \mathrm{d}\omega,\ j=1,\ldots,m,$
$\int_{\Omega_{t_0,t_1}} \left\langle \overline{\vartheta}_\alpha^\beta(\cdot), H_\beta^\alpha(t,\cdot,\cdot) - \frac{\partial x^\alpha}{\partial t^\beta}\right\rangle \mathrm{d}\omega,\ \beta=1,\ldots,q,\ \alpha=1,\ldots,n,$ are assumed to be invex functionals at $(\overline{x}(\cdot),\overline{u}(\cdot))$ on $\mathcal{X}\times\mathcal{U}$. Then, by Definition 4, the inequalities

$$\int_{\Omega_{t_0,t_1}} \left(\left\langle \overline{\mu}_j(t), G_j\left(t,\widetilde{x}(t),\widetilde{u}(t)\right)\right\rangle - \left\langle \overline{\mu}_j(t), G_j\left(t,\overline{x}(t),\overline{u}(t)\right)\right\rangle\right) \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \eta\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \left\langle \overline{\mu}_j(t), \frac{\partial G_j}{\partial x^\alpha}\left(t,\overline{x}(t),\overline{u}(t)\right)\right\rangle \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \xi\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \left\langle \overline{\mu}_j(t), \frac{\partial G_j}{\partial u^r}\left(t,\overline{x}(t),\overline{u}(t)\right)\right\rangle \mathrm{d}\omega, \tag{18}$$

$$\int\limits_{\Omega_{t_0,t_1}} \left(\left\langle \overline{\vartheta}_{\alpha}^{\beta}(t), H_{\beta}^{\alpha}\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) - \frac{\partial \widetilde{x}^{\alpha}}{\partial t^{\beta}} \right\rangle - \left\langle \overline{\vartheta}_{\alpha}^{\beta}(t), H_{\beta}^{\alpha}\left(t, \overline{x}(t), \overline{u}(t)\right) - \frac{\partial \overline{x}^{\alpha}}{\partial t^{\beta}} \right\rangle \right) \mathrm{d}\omega$$

$$\geqslant \int\limits_{\Omega_{t_0,t_1}} \eta\left(t, \widetilde{x}(t), \widetilde{u}(t), \overline{x}(t), \overline{u}(t)\right) \left\langle \overline{\vartheta}_{\alpha}^{\beta}(t), \frac{\partial H_{\beta}^{\alpha}}{\partial x^{\alpha}}\left(t, \overline{x}(t), \overline{u}(t)\right) \right\rangle \mathrm{d}\omega$$

$$- \int\limits_{\Omega_{t_0 t_1}} \overline{\vartheta}_{\alpha}^{\beta}(t) D_{\beta}\left(\eta\left(t, x(t), u(t), \overline{x}(t), \overline{u}(t)\right)\right) \mathrm{d}\omega$$

$$+ \int\limits_{\Omega_{t_0,t_1}} \xi\left(t, \widetilde{x}(t), \widetilde{u}(t), \overline{x}(t), \overline{u}(t)\right) \left\langle \overline{\vartheta}_{\alpha}^{\beta}(t), \frac{\partial H_{\beta}^{\alpha}}{\partial u^{r}}\left(t, \overline{x}(t), \overline{u}(t)\right) \right\rangle \mathrm{d}\omega \tag{19}$$

hold. By direct computations, we obtain

$$D_{\beta}\left[\eta \overline{\vartheta}_{\alpha}^{\beta}(t)\right] = \overline{\vartheta}_{\alpha}^{\beta}(t) D_{\beta}\eta + \eta D_{\beta}\overline{\vartheta}_{\alpha}^{\beta}(t). \tag{20}$$

Thus, (20) gives

$$\int\limits_{\Omega_{t_0 t_1}} \eta D_{\beta}\overline{\vartheta}_{\alpha}^{\beta}(t) \mathrm{d}\omega = \int\limits_{\Omega_{t_0 t_1}} D_{\beta}\left[\eta \overline{\vartheta}_{\alpha}^{\beta}(t)\right] \mathrm{d}\omega - \int\limits_{\Omega_{t_0 t_1}} \left[\overline{\vartheta}_{\alpha}^{\beta}(t) D_{\beta}\eta\right] \mathrm{d}\omega. \tag{21}$$

Using the flow-divergence formula, one has

$$\int\limits_{\Omega_{t_0 t_1}} D_{\beta}\left[\eta \overline{\vartheta}_{\alpha}^{\beta}(t)\right] \mathrm{d}\omega = \int\limits_{\partial\Omega_{t_0 t_1}} \left(\overline{\vartheta}_{\alpha}^{\beta}(t)\eta\right) \overrightarrow{n}(t) \mathrm{d}\sigma = 0, \tag{22}$$

where $\overrightarrow{n}(t) = (n_{\alpha}(t))$, $\alpha = 1, \ldots, n$, is the normal vector to the boundary $\partial\Omega_{t_0 t_1}$ and $\eta|_{\partial\Omega_{t_0,t_1}} = 0$. Combining (21) and (22), we obtain

$$\int\limits_{\Omega_{t_0 t_1}} \eta D_{\beta}\overline{\vartheta}_{\alpha}^{\beta}(t) \mathrm{d}\omega = - \int\limits_{\Omega_{t_0 t_1}} \left[\overline{\vartheta}_{\alpha}^{\beta}(t) D_{\beta}\eta\right] \mathrm{d}\omega. \tag{23}$$

Using (23) in (19), and then adding both sides of the resulting inequalities, (17) and (18), we get

$$\int\limits_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^{L} f^{L}\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) - \overline{\lambda}^{L} f^{L}\left(t, \overline{x}(t), \overline{u}(t)\right)\right) \mathrm{d}\omega$$

$$+ \int\limits_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^{U} f^{U}\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) - \overline{\lambda}^{U} f^{U}\left(t, \overline{x}(t), \overline{u}(t)\right)\right) \mathrm{d}\omega$$

$$
\begin{aligned}
&+\int\limits_{\Omega_{t_0,t_1}} \left(\left\langle \overline{\mu}_j(t), G_j\left(t,\widetilde{x}(t),\widetilde{u}(t)\right)\right\rangle - \left\langle \overline{\mu}_j(t), G_j\left(t,\overline{x}(t),\overline{u}(t)\right)\right\rangle \right) \mathrm{d}\omega \\
&+\int\limits_{\Omega_{t_0,t_1}} \left(\left\langle \overline{\vartheta}^{\beta}_{\alpha}(t), H^{\alpha}_{\beta}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \frac{\partial \widetilde{x}^{\alpha}}{\partial t^{\beta}} \right\rangle - \left\langle \overline{\vartheta}^{\beta}_{\alpha}(t), H^{\alpha}_{\beta}\left(t,\overline{x}(t),\overline{u}(t)\right) - \frac{\partial \overline{x}^{\alpha}}{\partial t^{\beta}} \right\rangle \right) \mathrm{d}\omega \\
&\geqslant \int\limits_{\Omega_{t_0,t_1}} \eta\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \left[\overline{\lambda}^{L} \frac{\partial f^{L}}{\partial x^{\alpha}}\left(\pi_{\overline{xu}}\right) + \overline{\lambda}^{U} \frac{\partial f^{U}}{\partial x^{\alpha}}\left(\pi_{\overline{xu}}\right) \right. \\
&\left. + \left\langle \overline{\mu}_j(t), \frac{\partial G_j}{\partial x^{\alpha}}\left(\pi_{\overline{xu}}\right)\right\rangle + \left\langle \overline{\vartheta}^{\beta}_{\alpha}(t), \frac{\partial H^{\alpha}_{\beta}}{\partial x^{\alpha}}\left(\pi_{\overline{xu}}\right)\right\rangle - \frac{\partial \overline{\vartheta}^{\beta}_{\alpha}(t)}{\partial t^{\beta}} \right] \mathrm{d}\omega \\
&+\int\limits_{\Omega_{t_0,t_1}} \xi\left(t,\widetilde{x}(t),\widetilde{u}(t),\overline{x}(t),\overline{u}(t)\right) \left[\overline{\lambda}^{L} \frac{\partial f^{L}}{\partial u^{r}}\left(\pi_{\overline{xu}}\right) + \overline{\lambda}^{U} \frac{\partial f^{U}}{\partial u^{r}}\left(\pi_{\overline{xu}}\right) \right. \\
&\left. + \left\langle \overline{\mu}_j(t), \frac{\partial G_j}{\partial u^{r}}\left(\pi_{\overline{xu}}\right)\right\rangle + \left\langle \overline{\vartheta}^{\beta}_{\alpha}(t), \frac{\partial H^{\alpha}_{\beta}}{\partial u^{r}}\left(\pi_{\overline{xu}}\right)\right\rangle \right] \mathrm{d}\omega.
\end{aligned}
$$

Hence, by the Karush-Kuhn-Tucker optimality conditions (6) and (7), the inequality above implies

$$
\begin{aligned}
&\int\limits_{\Omega_{t_0,t_1}} \overline{\lambda}^{L} f^{L}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega + \int\limits_{\Omega_{t_0,t_1}} \overline{\lambda}^{U} f^{U}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega \\
&\quad + \int\limits_{\Omega_{t_0,t_1}} \left\langle \overline{\mu}_j(t), G_j\left(t,\widetilde{x}(t),\widetilde{u}(t)\right)\right\rangle \mathrm{d}\omega + \int\limits_{\Omega_{t_0,t_1}} \left\langle \overline{\vartheta}^{\beta}_{\alpha}(t), H^{\alpha}_{\beta}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \frac{\partial \widetilde{x}^{\alpha}}{\partial t^{\beta}} \right\rangle \mathrm{d}\omega \\
&\geqslant \int\limits_{\Omega_{t_0,t_1}} \overline{\lambda}^{L} f^{L}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega + \int\limits_{\Omega_{t_0,t_1}} \overline{\lambda}^{U} f^{U}\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega \\
&\quad + \int\limits_{\Omega_{t_0,t_1}} \left\langle \overline{\mu}_j(t), G_j\left(t,\overline{x}(t),\overline{u}(t)\right)\right\rangle \mathrm{d}\omega + \int\limits_{\Omega_{t_0,t_1}} \left\langle \overline{\vartheta}^{\beta}_{\alpha}(t), H^{\alpha}_{\beta}\left(t,\overline{x}(t),\overline{u}(t)\right) - \frac{\partial \overline{x}^{\alpha}}{\partial t^{\beta}} \right\rangle \mathrm{d}\omega.
\end{aligned}
$$

Using the Karush-Kuhn-Tucker optimality condition (8) together with $(\overline{x}(\cdot),\overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$, we get

$$
\int\limits_{\Omega_{t_0,t_1}} \overline{\lambda}^{L} f^{L}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega + \int\limits_{\Omega_{t_0,t_1}} \overline{\lambda}^{U} f^{U}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega
$$

$$+\int_{\Omega_{t_0,t_1}} \left\langle \overline{\mu}_j(t), G_j\left(t,\widetilde{x}(t),\widetilde{u}(t)\right)\right\rangle \mathrm{d}\omega + \int_{\Omega_{t_0,t_1}} \left\langle \overline{\vartheta}^{\beta}_{\alpha}(t), H^{\alpha}_{\beta}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \frac{\partial \widetilde{x}^{\alpha}}{\partial t^{\beta}}\right\rangle \mathrm{d}\omega$$

$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega + \int_{\Omega_{t_0,t_1}} \overline{\lambda}^U f^U\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega.$$

Hence, by (12), it follows that

$$\int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega + \int_{\Omega_{t_0,t_1}} \overline{\lambda}^U f^U\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega$$
$$+\int_{\Omega_{t_0,t_1}} \sum_{j=1}^{m} \overline{\mu}_j(t) G^{+}_j\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega$$
$$+\int_{\Omega_{t_0,t_1}} \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q} \left|\overline{\vartheta}^{\beta}_{\alpha}(t)\right| \left| H^{\alpha}_{\beta}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \frac{\partial \widetilde{x}^{\alpha}}{\partial t^{\beta}}\right| \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega + \int_{\Omega_{t_0,t_1}} \overline{\lambda}^U f^U\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega. \tag{24}$$

i) Now, we consider the case when $\sum_{j=1}^{m} \overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q} \left|\overline{\vartheta}^{\beta}_{\alpha}(\cdot)\right| > 0.$

Let us denote $\overline{c}(\cdot) = \sum_{j=1}^{m} \overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q} \left|\overline{\vartheta}^{\beta}_{\alpha}(\cdot)\right|$. Thus,

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t,\widetilde{x}(t),\widetilde{u}(t)\right)\right) \mathrm{d}\omega$$
$$+\int_{\Omega_{t_0,t_1}} \overline{c}(t) \sum_{j=1}^{m} \frac{\overline{\mu}_j(t)}{\overline{c}(t)} G^{+}_j\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega$$
$$+\int_{\Omega_{t_0,t_1}} \overline{c}(t) \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q} \frac{\left|\overline{\vartheta}^{\beta}_{\alpha}(t)\right|}{\overline{c}(t)} \left| H^{\alpha}_{\beta}\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \frac{\partial \widetilde{x}^{\alpha}}{\partial t^{\beta}}\right| \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega. \tag{25}$$

Let us denote

$$\overline{\theta}_l(\cdot) = \frac{\overline{\mu}_l(\cdot)}{\overline{c}(\cdot)}, \quad l = 1, \ldots, m, \tag{26}$$

$$\overline{\theta}_{m+l+k}(\cdot) = \frac{\left|\overline{\vartheta}^l_k(\cdot)\right|}{\overline{c}(\cdot)}, \quad l = 1, \ldots, n, \ k = 1, \ldots, q, \tag{27}$$

$$\psi_l\left(t, \widetilde{x}(\cdot), \widetilde{u}(\cdot)\right) = G^+_l\left(t, \widetilde{x}(\cdot), \widetilde{u}(\cdot)\right), \quad l = 1, \ldots, m, \tag{28}$$

$$\psi_{m+l+k}\left(t, \widetilde{x}(\cdot), \widetilde{u}(\cdot)\right) = \left|H^l_k\left(t, \widetilde{x}(\cdot), \widetilde{u}(\cdot)\right) - \frac{\partial \widetilde{x}^l}{\partial t^k}\right|, \ l = 1, \ldots, n, \ k = 1, ...q. \tag{29}$$

Then, note that $\overline{\theta}_l \geqslant 0$, $l = 1, \ldots, m+nq$, and, by $\overline{c}(\cdot) = \sum_{j=1}^{m} \overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}\left|\overline{\vartheta}^\beta_\alpha(\cdot)\right|$, it follows that $\sum_{l=1}^{m+nq} \overline{\theta}_l = 1$. Using (26)–(29) in (25), we get

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t, \widetilde{x}(t), \widetilde{u}(t)\right)\right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \overline{c}(t) \sum_{l=1}^{m+nq} \overline{\theta}_l(t)\psi_l\left(t, \widetilde{x}(t), \widetilde{u}(t)\right)$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega. \tag{30}$$

Thus, (30) gives

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t, \widetilde{x}(t), \widetilde{u}(t)\right)\right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \overline{c}(t) \max_{\theta_l(t)\in\Delta(t)} \sum_{l=1}^{m+nq} \theta_l(t)\psi_l\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega,$$

where $\Delta(t) = \left\{ \theta(t) \in R_{+}^{m+nq} : \sum_{l=1}^{m+nq} \theta_l(t) = 1 \right\}$. Hence, by Lemma 1, we get

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L(t, \widetilde{x}(t), \widetilde{u}(t))\, \mathrm{d}\omega + \overline{\lambda}^U f^U(t, \widetilde{x}(t), \widetilde{u}(t)) \right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \overline{c}(t) \max_{1 \leqslant l \leqslant m+nq} \psi_l(t, \widetilde{x}(t), \widetilde{u}(t))\, \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L(t, \overline{x}(t), \overline{u}(t))\, \mathrm{d}\omega + \overline{\lambda}^U f^U(t, \overline{x}(t), \overline{u}(t))\, \mathrm{d}\omega. \tag{31}$$

Again using (26)–(29) in (31), we get

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L(t, \widetilde{x}(t), \widetilde{u}(t))\, \mathrm{d}\omega + \overline{\lambda}^U f^U(t, \widetilde{x}(t), \widetilde{u}(t)) \right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \overline{c}(t) \max_{\substack{1 \leqslant \beta \leqslant q \\ 1 \leqslant \alpha \leqslant n \\ 1 \leqslant j \leqslant m}} \left\{ G_j^+(t, \widetilde{x}(t), \widetilde{u}(t)), \left| H_\beta^\alpha(t, \widetilde{x}(t), \widetilde{u}(t)) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \right| \right\}$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L(t, \overline{x}(t), \overline{u}(t))\, \mathrm{d}\omega + \overline{\lambda}^U f^U(t, \overline{x}(t), \overline{u}(t))\, \mathrm{d}\omega. \tag{32}$$

Using the assumption $c(\cdot) \geqslant \overline{c}(\cdot) = \sum_{j=1}^{m} \overline{\mu}_j(t\cdot) + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{q} \left| \overline{\vartheta}_\alpha^\beta(\cdot) \right|$, we get that the inequality

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L(t, \widetilde{x}(t), \widetilde{u}(t))\, \mathrm{d}\omega + \overline{\lambda}^U f^U(t, \widetilde{x}(t), \widetilde{u}(t)) \right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} c(t) \max_{\substack{1 \leqslant \beta \leqslant q \\ 1 \leqslant \alpha \leqslant n \\ 1 \leqslant j \leqslant m}} \left\{ G_j^+(t, \widetilde{x}(t), \widetilde{u}(t)), \left| H_\beta^\alpha(t, \widetilde{x}(t), \widetilde{u}(t)) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta} \right| \right\}$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L(t, \overline{x}(t), \overline{u}(t))\, \mathrm{d}\omega + \overline{\lambda}^U f^U(t, \overline{x}(t), \overline{u}(t))\, \mathrm{d}\omega$$

holds, which contradicting (14).

ii) Now, we consider the case when $\sum_{j=1}^{m} \overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q} \left|\overline{\vartheta}_{\alpha}^{\beta}(\cdot)\right| = 0.$

Then, (24) yields

$$\int\limits_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) + \overline{\lambda}^U f^U\left(t,\widetilde{x}(t),\widetilde{u}(t)\right)\right) \mathrm{d}\omega$$
$$\geqslant \int\limits_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(t,\overline{x}(t),\overline{u}(t)\right) + \overline{\lambda}^U f^U\left(t,\overline{x}(t),\overline{u}(t)\right)\right) \mathrm{d}\omega. \tag{33}$$

Thus, (33) implies that the inequality

$$\int\limits_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t,\widetilde{x}(t),\widetilde{u}(t)\right)\right) \mathrm{d}\omega$$
$$+ \int\limits_{\Omega_{t_0,t_1}} c(t) \max_{\substack{1\leqslant\beta\leqslant q\\ 1\leqslant\alpha\leqslant n\\ 1\leqslant j\leqslant m}} \left\{G_j^+\left(t,\widetilde{x}(t),\widetilde{u}(t)\right), \left|H_\beta^\alpha\left(t,\widetilde{x}(t),\widetilde{u}(t)\right) - \frac{\partial \widetilde{x}^\alpha}{\partial t^\beta}\right|\right\}$$
$$\geqslant \int\limits_{\Omega_{t_0,t_1}} \overline{\lambda}^L f^L\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega + \overline{\lambda}^U f^U\left(t,\overline{x}(t),\overline{u}(t)\right) \mathrm{d}\omega$$

holds, which contradicting (14).

This means that $(\overline{x}(\cdot),\overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ is an LU-minimizer in (IVMCP$_\infty(c)$)) and completes the proof of this theorem.

This corollary follows directly from Theorem 2

Corollary 1. *Let $(\overline{x}(\cdot),\overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ be a normal LU-optimal solution in the considered interval-valued multitime control problem (IVMCP) with first-order PDE constraints. Further, assume that all the hypotheses of Theorem 2 are fulfilled. Then $(\overline{x}(\cdot),\overline{u}(\cdot))$ is a LU-minimizer in (IVMCP$_\infty(c)$).*

Now, we prove the converse result to that one established in Theorem 2

Theorem 3. *Let $(\overline{x}(\cdot),\overline{u}(\cdot))$ be a normal LU-optimal solution in the unconstrained penalized multitime variational problem (IVMCP$_\infty(c)$) with the interval-valued minimax exact penalty function. Further, assume that:*

a) $\int\limits_{\Omega_{t_0,t_1}} f(t,\cdot,\cdot)\mathrm{d}\omega$ *is an invex interval-valued functional on $\Gamma\left(\Omega_{t_0,t_1}\right)$ with respect η and ξ,* *b)* $\int\limits_{\Omega_{t_0,t_1}} \left\langle \widehat{\mu}_j(\cdot), G_j(t,\cdot,\cdot) \right\rangle \mathrm{d}\omega$*, $j = 1, \ldots, m$, are invex functionals on* $\Gamma\left(\Omega_{t_0,t_1}\right)$ *with respect η and ξ,*

c) $\int\limits_{\Omega_{t_0,t_1}} \left\langle \widehat{\vartheta}^{\beta}_{\alpha}(\cdot), H^{\alpha}_{\beta}(t,\cdot,\cdot) - \dfrac{\partial \widehat{x}^{\alpha}}{\partial t^{\beta}} \right\rangle \mathrm{d}\omega$*, $\beta = 1, \ldots, q$, $\alpha = 1, \ldots, n$, are invex functionals on* $\Gamma\left(\Omega_{t_0,t_1}\right)$ *with respect η and ξ,*

where $\left(\left(\widehat{x}(\cdot), \widehat{u}(\cdot)\right), \widehat{\lambda}^L, \widehat{\lambda}^U, \widehat{\mu}(\cdot), \widehat{\vartheta}(\cdot)\right)$ *is any Karush-Kuhn-Tucker point in (IVMCP) with $\widehat{\lambda}^L > 0$ and $\widehat{\lambda}^U > 0$. If the original interval-valued multitime control problem (IVMCP) with first-order PDE constraints is solvable and the penalty parameter $c(\cdot)$ is assumed to be sufficiently large (that is, it is sufficient to assume that* $c(\cdot) \geqslant \sum_{j=1}^{m} \widehat{\mu}_j(\cdot) + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{q} \left|\widehat{\vartheta}^{\beta}_{\alpha}(\cdot)\right|$*), then $(\overline{x}(\cdot), \overline{u}(\cdot))$ is also a LU-minimizer in (IVMCP).*

Proof. Assume that $(\overline{x}(\cdot), \overline{u}(\cdot))$ is an LU-optimal solution in the unconstrained interval-valued penalized multitime variational problem (IVMCP$_{\infty}(c)$). We proceed by contradiction. Suppose, contrary to the result, that $(\overline{x}(\cdot), \overline{u}(\cdot))$ is also a LU-minimizer in (IVMCP). By assumption, the original interval-valued multitime control problem (IVMCP) with first-order PDE constraints is solvable. Then, by Definition 3, there exists at least one LU-optimal solution in (IVMCP). We denote it by $(\widetilde{x}(\cdot), \widetilde{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$. Then, by Definition 2, one has

$$\int\limits_{\Omega_{t_0,t_1}} f\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) \mathrm{d}\omega \prec_{LU} \int\limits_{\Omega_{t_0,t_1}} f\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega.$$

Then, by the definition of the relation $\prec_{LU}$, we have

$$\left(\int\limits_{\Omega_{t_0,t_1}} f^L\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega > \int\limits_{\Omega_{t_0,t_1}} f^L\left(\pi_{\widetilde{xu}}\right) \mathrm{d}\omega \ \wedge \int\limits_{\Omega_{t_0,t_1}} f^U\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \geqslant \int\limits_{\Omega_{t_0,t_1}} f^U\left(\pi_{\widetilde{xu}}\right) \mathrm{d}\omega\right)$$

or

$$\left(\int\limits_{\Omega_{t_0,t_1}} f^L\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \geqslant \int\limits_{\Omega_{t_0,t_1}} f^L\left(\pi_{\widetilde{xu}}\right) \mathrm{d}\omega \ \wedge \int\limits_{\Omega_{t_0,t_1}} f^U\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega > \int\limits_{\Omega_{t_0,t_1}} f^U\left(\pi_{\widetilde{xu}}\right) \mathrm{d}\omega\right)$$

or

$$\left(\int_{\Omega_{t_0,t_1}} f^L\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} f^L\left(\pi_{\widetilde{xu}}\right) \mathrm{d}\omega \ \wedge \int_{\Omega_{t_0,t_1}} f^U\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} f^U\left(\pi_{\widetilde{xu}}\right) \mathrm{d}\omega \right).$$

Multiplying the above inequalities by the corresponding Lagrange multipliers $\overline{\lambda}^L > 0, \overline{\lambda}^U > 0$ associated to the endpoint functionals f^L and f^U of the interval-valued objective function and then adding the resulting inequalities, we get

$$\int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(\pi_{\overline{xu}}\right) + \overline{\lambda}^U f^U\left(\pi_{\overline{xu}}\right)\right) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} \left(\overline{\lambda}^L f^L\left(\pi_{\widetilde{xu}}\right) + \overline{\lambda}^U f^U\left(\pi_{\widetilde{xu}}\right)\right) \mathrm{d}\omega. \quad (34)$$

Since $(\overline{x}(\cdot), \overline{u}(\cdot))$ is an LU-optimal solution in the unconstrained penalized multitime variational problem (IVMCP$_\infty(c)$), by Definition 2, the following relation

$$P_\infty\left(\left(\overline{x}(\cdot), \overline{u}(\cdot)\right), c(\cdot)\right) \prec_{LU} P_\infty\left(\left(x(\cdot), u(\cdot)\right), c(\cdot)\right)$$

holds for all $(x(\cdot), u(\cdot)) \in \mathcal{X} \times \mathcal{U}$. Thus, by the definition of the relation $\prec_{LU}$, the aforesaid inequality yields, for all $(x(\cdot), u(\cdot)) \in \mathcal{X} \times \mathcal{U}$,

$$\left(\int_{\Omega_{t_0,t_1}} P_\infty^L\left(\left(x(t), u(t)\right), c(t)\right) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^L\left(\left(\overline{x}(t), \overline{u}(t)\right), c(t)\right) \mathrm{d}\omega \ \wedge \right.$$

$$\left. \int_{\Omega_{t_0,t_1}} P_\infty^U\left(\left(x(t), u(t)\right), c(t)\right) \mathrm{d}\omega \geqslant \int_{\Omega_{t_0,t_1}} P_\infty^U\left(\left(\overline{x}(t), \overline{u}(t)\right), c(t)\right) \mathrm{d}\omega \right)$$

or

$$\left(\int_{\Omega_{t_0,t_1}} P_\infty^L\left(\left(x(t), u(t)\right), c(t)\right) \mathrm{d}\omega \geqslant \int_{\Omega_{t_0,t_1}} P_\infty^L\left(\left(\overline{x}(t), \overline{u}(t)\right), c(t)\right) \mathrm{d}\omega \ \wedge \right.$$

$$\left. \int_{\Omega_{t_0,t_1}} P_\infty^U\left(\left(x(t), u(t)\right), c(t)\right) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^U\left(\left(\overline{x}(t), \overline{u}(t)\right), c(t)\right) \mathrm{d}\omega \right)$$

or

$$\left(\int_{\Omega_{t_0,t_1}} P_\infty^L\left(\left(x(t), u(t)\right), c(t)\right) \mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^L\left(\left(\overline{x}(t), \overline{u}(t)\right), c(t)\right) \mathrm{d}\omega \ \wedge \right.$$

$$\int_{\Omega_{t_0,t_1}} P_\infty^U\left(\left(x(t),u(t)\right),c(t)\right)\mathrm{d}\omega > \int_{\Omega_{t_0,t_1}} P_\infty^U\left(\left(\overline{x}(t),\overline{u}(t)\right),c(t)\right)\mathrm{d}\omega\Bigg).$$

Thus, by the definition of of the interval-valued minimax penalty functional P_∞, one has for all $(x(\cdot),u(\cdot)) \in \mathcal{X}\times\mathcal{U}$,

$$\Bigg(\int_{\Omega_{t_0,t_1}}\left(f^L(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega$$

$$> \int_{\Omega_{t_0,t_1}}\left(f^L(\pi_{\overline{xu}}) + c(t)\max\left\{G_j^+(\pi_{\overline{xu}}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{\overline{xu}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega \wedge$$

$$\Bigg(\int_{\Omega_{t_0,t_1}}\left(f^U(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega$$

$$\geqslant \int_{\Omega_{t_0,t_1}}\left(f^U(\pi_{\overline{xu}}) + c(t)\max\left\{G_j^+(\pi_{\overline{xu}}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{\overline{xu}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega\Bigg)$$

or

$$\Bigg(\int_{\Omega_{t_0,t_1}}\left(f^L(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega$$

$$\geqslant \int_{\Omega_{t_0,t_1}}\left(f^L(\pi_{\overline{xu}}) + c(t)\max\left\{G_j^+(\pi_{\overline{xu}}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{\overline{xu}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega \wedge$$

$$\Bigg(\int_{\Omega_{t_0,t_1}}\left(f^U(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega$$

$$> \int_{\Omega_{t_0,t_1}}\left(f^U(\pi_{\overline{xu}}) + c(t)\max\left\{G_j^+(\pi_{\overline{xu}}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{\overline{xu}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right)\mathrm{d}\omega\Bigg)$$

or

$$\left(\int_{\Omega_{t_0,t_1}} \left(f^L(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right) \mathrm{d}\omega\right.$$

$$> \int_{\Omega_{t_0,t_1}} \left(f^L(\pi_{\overline{xu}}) + c(t)\max\left\{G_j^+(\pi_{\overline{xu}}),\, j\in J, \left|H_\beta^\alpha(\pi_{\overline{xu}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right) \mathrm{d}\omega \ \wedge$$

$$\int_{\Omega_{t_0,t_1}} \left(f^U(\pi_{xu}) + c\max\left\{G_j^+(\pi_{xu}),\, j\in J, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right) \mathrm{d}\omega$$

$$\left. > \int_{\Omega_{t_0,t_1}} \left(f^U(\pi_{\overline{xu}}) + c(t)\max\left\{G_j^+(\pi_{\overline{xu}}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{\overline{xu}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right) \mathrm{d}\omega\right).$$

Then, by (12), the aforesaid inequalities yield for all $(x(\cdot), u(\cdot)) \in \mathcal{X}\times\mathcal{U}$,

$$\left(\int_{\Omega_{t_0,t_1}} \left(f^L(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right) \mathrm{d}\omega\right.$$

$$> \int_{\Omega_{t_0,t_1}} f^L\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \ \wedge$$

$$\int_{\Omega_{t_0,t_1}} \left(f^U(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right) \mathrm{d}\omega$$

$$\left. \geqslant \int_{\Omega_{t_0,t_1}} f^U\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega\right)$$

or

$$\left(\int_{\Omega_{t_0,t_1}} \left(f^L(\pi_{xu}) + c(t)\max\left\{G_j^+(\pi_{xu}),\, j\in J,\, \left|H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta}\right|, \alpha\in A, \beta\in B\right\}\right) \mathrm{d}\omega\right.$$

$$\geqslant \int_{\Omega_{t_0,t_1}} f^L\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \ \wedge$$

$$\int\limits_{\Omega_{t_0,t_1}} \left(f^U(\pi_{xu}) + c(t) \max \left\{ G_j^+(\pi_{xu}),\ j \in J,\ \left| H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta} \right|, \alpha \in A, \beta \in B \right\} \right) \mathrm{d}\omega$$

$$\left. > \int\limits_{\Omega_{t_0,t_1}} f^U\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \right)$$

or

$$\left(\int\limits_{\Omega_{t_0,t_1}} \left(f^L(\pi_{xu}) + c(t) \max \left\{ G_j^+(\pi_{xu}),\ j \in J, \left| H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta} \right|, \alpha \in A, \beta \in B \right\} \right) \mathrm{d}\omega \right.$$

$$> \int\limits_{\Omega_{t_0,t_1}} f^L\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \ \wedge$$

$$\int\limits_{\Omega_{t_0,t_1}} \left(f^U(\pi_{xu}) + c \max \left\{ G_j^+(\pi_{xu}),\ j \in J, \left| H_\beta^\alpha(\pi_{xu}) - \frac{\partial x^\alpha}{\partial t^\beta} \right|, \alpha \in A, \beta \in B \right\} \right) \mathrm{d}\omega$$

$$\left. > \int\limits_{\Omega_{t_0,t_1}} f^U\left(\pi_{\overline{xu}}\right) \mathrm{d}\omega \right).$$

Since the above inequalities are satisfied for all $(x(\cdot), u(\cdot)) \in \mathcal{X} \times \mathcal{U}$, they are also fulfilled for all $(x(\cdot), u(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$. Then, using (12) again, the following inequalities

$$\begin{gathered}
\left(\int\limits_{\Omega_{t_0,t_1}} \left(f^L(\pi_{xu}) - f^L\left(\pi_{\overline{xu}}\right) \right) \mathrm{d}\omega > 0 \ \wedge \int\limits_{\Omega_{t_0,t_1}} \left(f^U(\pi_{xu}) - f^U\left(\pi_{\overline{xu}}\right) \right) \mathrm{d}\omega \geqslant 0 \right) \\
\left(\int\limits_{\Omega_{t_0,t_1}} \left(f^L(\pi_{xu}) - f^L\left(\pi_{\overline{xu}}\right) \right) \mathrm{d}\omega \geqslant 0 \ \wedge \int\limits_{\Omega_{t_0,t_1}} \left(f^U(\pi_{xu}) - f^U\left(\pi_{\overline{xu}}\right) \right) \mathrm{d}\omega > 0 \right) \\
\left(\int\limits_{\Omega_{t_0,t_1}} \left(f^L(\pi_{xu}) - f^L\left(\pi_{\overline{xu}}\right) \right) \mathrm{d}\omega > 0 \ \wedge \int\limits_{\Omega_{t_0,t_1}} \left(f^U(\pi_{xu}) - f^U\left(\pi_{\overline{xu}}\right) \right) \mathrm{d}\omega > 0 \right)
\end{gathered} \tag{35}$$

hold for all $(x(\cdot), u(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$. Then, by the definition of the relation $\prec_{LU}$, (35) implies that the relation

$$\int\limits_{\Omega_{t_0,t_1}} f\left(t, \overline{x}(t), \overline{u}(t)\right) \mathrm{d}\omega \prec_{LU} \int\limits_{\Omega_{t_0,t_1}} f\left(t, x(t), u(t)\right) \mathrm{d}\omega \tag{36}$$

holds for all $(x(\cdot), u(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$.

Hence, in order to show $(\overline{x}(\cdot), \overline{u}(\cdot))$ is an LU-optimal solution of (IVMCP), we have to prove that it is feasible in (IVMCP). Since $(\widehat{x}(\cdot), \widehat{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$ is an LU-optimal solution of (IVMCP). there exist $\widehat{\lambda}^L \in \mathbb{R}_+$, $\widehat{\lambda}^U \in \mathbb{R}_+$ and piecewise smooth functions $\widehat{\mu}\colon \Omega_{t_0,t_1} \to \mathbb{R}^m_+$, $\overline{\vartheta}\colon \Omega_{t_0,t_1} \to \mathbb{R}^{qn}$ such that the necessary optimality conditions (6)–(9) are fulfilled for all $t \in \Omega_{t_0,t_1}$, except at discontinuities. Moreover, by Definition 7, $\left((\widehat{x}(\cdot), \widehat{u}(\cdot)), \widehat{\lambda}^L, \widehat{\lambda}^U, \widehat{\mu}(\cdot), \widehat{\vartheta}(\cdot)\right)$ is a Karush-Kuhn-Tucker point in (IVMCP) with $\widehat{\lambda}^L > 0$ and $\widehat{\lambda}^U > 0$. Hence, by invexity assumptions, the following inequalities

$$\begin{aligned}
&\int\limits_{\Omega_{t_0,t_1}} \left(f^L\left(t, \overline{x}(t), \overline{u}(t)\right) - f^L\left(t, \widehat{x}(t), \widehat{u}(t)\right)\right) \mathrm{d}\omega \\
&\qquad \geqslant \int\limits_{\Omega_{t_0,t_1}} \eta\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \frac{\partial f^L}{\partial x^\alpha}\left(t, \widehat{x}(t), \widehat{u}(t)\right) \mathrm{d}\omega \\
&\qquad\quad + \int\limits_{\Omega_{t_0,t_1}} \xi\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \frac{\partial f^L}{\partial u^r}\left(t, \widehat{x}(t), \widehat{u}(t)\right) \mathrm{d}\omega,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\int\limits_{\Omega_{t_0,t_1}} \left(f^U\left(t, \widetilde{x}(t), \widetilde{u}(t)\right) - f^U\left(t, \widehat{x}(t), \widehat{u}(t)\right)\right) \mathrm{d}\omega \\
&\qquad \geqslant \int\limits_{\Omega_{t_0,t_1}} \eta\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \frac{\partial f^U}{\partial x^\alpha}\left(t, \widehat{x}(t), \widehat{u}(t)\right) \mathrm{d}\omega \\
&\qquad\quad + \int\limits_{\Omega_{t_0,t_1}} \xi\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \frac{\partial f^U}{\partial u^r}\left(t, \widehat{x}(t), \widehat{u}(t)\right) \mathrm{d}\omega,
\end{aligned} \tag{38}$$

$$\int_{\Omega_{t_0,t_1}} \left(\left\langle \widehat{\mu}_j(t), G_j\left(t, \overline{x}(t), \overline{u}(t)\right)\right\rangle - \left\langle \widehat{\mu}_j(t), G_j\left(t, \widehat{x}(t), \widehat{u}(t)\right)\right\rangle\right) \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \eta\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \left\langle \widehat{\mu}_j(t), \frac{\partial G_j}{\partial x^\alpha}\left(t, \widehat{x}(t), \widehat{u}(t)\right)\right\rangle \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \xi\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \left\langle \widehat{\mu}_j(t), \frac{\partial G_j}{\partial u^r}\left(t, \widehat{x}(t), \widehat{u}(t)\right)\right\rangle \mathrm{d}\omega, \quad (39)$$

$$\int_{\Omega_{t_0,t_1}} \left(\left\langle \widehat{\vartheta}_\alpha^\beta(t), H_\beta^\alpha\left(t, \overline{x}(t), \overline{u}(t)\right) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right\rangle - \left\langle \widehat{\vartheta}_\alpha^\beta(t), H_\beta^\alpha\left(t, \widehat{x}(t), \widehat{u}(t)\right) - \frac{\partial \widehat{x}^\alpha(t)}{\partial t^\beta}\right\rangle\right) \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \eta\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \left(\left\langle \widehat{\vartheta}_\alpha^\beta(t), \frac{\partial H_\beta^\alpha}{\partial x^\alpha}\left(t, \widehat{x}(t), \widehat{u}(t)\right)\right\rangle\right) \mathrm{d}\omega$$
$$- \int_{\Omega_{t_0,t_1}} \widehat{\vartheta}_\alpha^\beta(t) D_\beta\left(\eta\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right)\right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} \xi\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \left\langle \widehat{\vartheta}_\beta^\alpha(t), \frac{\partial H_\beta}{\partial u^r}\left(t, \widehat{x}(t), \widehat{u}(t)\right)\right\rangle \mathrm{d}\omega \quad (40)$$

are satisfied. Multiplying (37) and (38) by $\widehat{\lambda}^L > 0$, $\widehat{\lambda}^U > 0$, respectively, and then summarizing both sides of the resulting inequalities together with (39) and (40), we get

$$
\begin{aligned}
&\int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L\left(\pi_{\overline{xu}}\right) - \widehat{\lambda}^L f^L\left(\pi_{\widehat{xu}}\right)\right) \mathrm{d}\omega + \int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^U f^U\left(\pi_{\overline{xu}}\right) - \widehat{\lambda}^U f^U\left(\pi_{\widehat{xu}}\right)\right) \mathrm{d}\omega \\
&\quad + \int_{\Omega_{t_0,t_1}} \left(\left\langle \widehat{\mu}_j(t), G_j\left(\pi_{\overline{xu}}\right)\right\rangle - \left\langle \widehat{\mu}_j(t), G_j\left(\pi_{\widehat{xu}}\right)\right\rangle\right) \mathrm{d}\omega \\
&\quad + \int_{\Omega_{t_0,t_1}} \left(\left\langle \widehat{\vartheta}_\beta^\alpha(t), H_\beta^\alpha\left(\pi_{\overline{xu}}\right) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right\rangle \mathrm{d}\omega - \left\langle \widehat{\vartheta}_\alpha^\beta(t), H_\beta^\alpha\left(\pi_{\widehat{xu}}\right) - \frac{\partial \widehat{x}^\alpha}{\partial t^\beta}\right\rangle\right) \mathrm{d}\omega \\
&\geqslant \int_{\Omega_{t_0,t_1}} \eta\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \left[\widehat{\lambda}^L \frac{\partial f^L}{\partial x^\alpha}\left(\pi_{\widehat{xu}}\right) + \widehat{\lambda}^U \frac{\partial f^U}{\partial x^\alpha}\left(\pi_{\widehat{xu}}\right)\right. \\
&\quad \left. + \left\langle \widehat{\mu}_j(t), \frac{\partial G_j}{\partial x^\alpha}\left(\pi_{\widehat{xu}}\right)\right\rangle + \left\langle \widehat{\vartheta}_\alpha^\beta(t), \frac{\partial H_\beta^\alpha}{\partial x^\alpha}\left(\pi_{\widehat{xu}}\right)\right\rangle - \frac{\partial \widehat{\vartheta}_\alpha^\beta(t)}{\partial t^\beta}\right] \mathrm{d}\omega \\
&\quad + \int_{\Omega_{t_0,t_1}} \xi\left(t, \overline{x}(t), \overline{u}(t), \widehat{x}(t), \widehat{u}(t)\right) \left[\widehat{\lambda}^L \frac{\partial f^L}{\partial u^r}\left(\pi_{\widehat{xu}}\right) + \widehat{\lambda}^U \frac{\partial f^U}{\partial u^r}\left(\pi_{\widehat{xu}}\right)\right. \\
&\quad \left. + \left\langle \widehat{\mu}_j(t), \frac{\partial G_j}{\partial u^r}\left(\pi_{\widehat{xu}}\right)\right\rangle + \left\langle \widehat{\vartheta}_\alpha^\beta(t), \frac{\partial H_\beta^\alpha}{\partial u^r}\left(\pi_{\widehat{xu}}\right)\right\rangle\right] \mathrm{d}\omega. \qquad (41)
\end{aligned}
$$

Since $\left(\left(\widehat{x}(\cdot), \widehat{u}(\cdot)\right), \widehat{\lambda}^L, \widehat{\lambda}^U, \widehat{\mu}(\cdot), \widehat{\xi}(\cdot)\right)$ is a Karush-Kuhn-Tucker point in (IVMCP), by Definition 7, the necessary optimality conditions (6)–(9) are fulfilled at $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ for all $t \in \Omega_{t_0,t_1}$, except at discontinuities. Hence, by the necessary optimality condition (6), we have

$$
\begin{aligned}
&\int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L(\pi_{\overline{xu}}) + \widehat{\lambda}^U f^U(\pi_{\overline{xu}}) + \left\langle \widehat{\mu}_j(t), G_j\left(\pi_{\overline{xu}}\right)\right\rangle + \left\langle \widehat{\vartheta}_\alpha^\beta(t), H_\beta^\alpha(\pi_{\overline{xu}}) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right\rangle\right) \mathrm{d}\omega \\
&\quad \geqslant \int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L(\pi_{\widehat{xu}}) + \widehat{\lambda}^U f^U(\pi_{\widehat{xu}}) + \left\langle \widehat{\mu}_j(t), G_j(\pi_{\widehat{xu}})\right\rangle\right. \\
&\quad \left. + \left\langle \widehat{\vartheta}_\alpha^\beta(t), H_\beta^\alpha\left(\pi_{\widehat{xu}}\right) - \frac{\partial \widehat{x}^\alpha}{\partial t^\beta}\right\rangle\right) \mathrm{d}\omega.
\end{aligned}
$$

Using (12) together with $(\widehat{x}(\cdot), \widehat{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$, we get

$$\int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L\left(\pi_{\overline{xu}}\right) + \widehat{\lambda}^U f^U\left(\pi_{\overline{xu}}\right) + \sum_{j=1}^{m} \widehat{\mu}_j(t) G_j^+\left(pi_{\overline{xu}}\right) \mathrm{d}\omega \right.$$
$$\left. + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{q} \left|\widehat{\vartheta}_\alpha^\beta(t)\right| \left| H_\beta^\alpha\left(\pi_{\overline{xu}}\right) - \frac{\partial x_\beta^\alpha}{\partial t^\beta} \right| \right) \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L\left(\pi_{\widehat{xu}}\right) + \widehat{\lambda}^U f^U\left(\pi_{\widehat{xu}}\right) \right) \mathrm{d}\omega. \tag{42}$$

In the similar way as in the proof of Theorem 2, we obtain that the inequality

$$\int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L\left(\pi_{\overline{xu}}\right) + \widehat{\lambda}^U f^U\left(\pi_{\overline{xu}}\right) \right) \mathrm{d}\omega$$
$$+ \int_{\Omega_{t_0,t_1}} c(t) \max_{\substack{1\leqslant\beta\leqslant n \\ 1\leqslant\alpha\leqslant n \\ 1\leqslant j\leqslant m}} \left\{ G_j^+\left(\pi_{\overline{xu}}\right), \left| H_\beta^\alpha\left(\pi_{\overline{xu}}\right) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta} \right| \right\} \mathrm{d}\omega$$
$$\geqslant \int_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L\left(\pi_{\widehat{xu}}\right) + \widehat{\lambda}^U f^U\left(\pi_{\widehat{xu}}\right) \right) \mathrm{d}\omega \tag{43}$$

is satisfied, where $c(\cdot) \geqslant \sum_{j=1}^{m} \widehat{\mu}_j(\cdot) + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{q} \left|\widehat{\vartheta}_\alpha^\beta(\cdot)\right|$. Since $(\widehat{x}(\cdot), \widehat{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$, by (12), it follows that

$$c(\cdot) \max \left\{ G_j^+\left(\pi_{\widehat{xu}}\right),\ j \in J,\ \left| H_\beta^\alpha\left(\pi_{\widehat{xu}}\right) - \frac{\partial \widehat{x}^\alpha}{\partial t^\beta} \right|,\ \alpha \in A,\ \beta \in B \right\} = 0. \tag{44}$$

Combining (43) and (44), we obtain

$$\begin{aligned}
&\int\limits_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L\left(\pi_{\overline{xu}}\right) + \widehat{\lambda}^U f^U\left(\pi_{\overline{xu}}\right)\right) \mathrm{d}\omega \\
&\qquad + \int\limits_{\Omega_{t_0,t_1}} c(t) \max_{\substack{1\leqslant\beta\leqslant q\\ 1\leqslant\alpha\leqslant n\\ 1\leqslant j\leqslant m}} \left\{G_j^+\left(\pi_{\overline{xu}}\right), \left|H_\beta^\alpha\left(\pi_{\overline{xu}}\right) - \frac{\partial \overline{x}^\alpha}{\partial t^\beta}\right|\right\} \mathrm{d}\omega \\
&\quad \geqslant \int\limits_{\Omega_{t_0,t_1}} \left(\widehat{\lambda}^L f^L\left(\pi_{\widehat{xu}}\right) + \widehat{\lambda}^U f^U\left(\pi_{\widehat{xu}}\right)\right) \mathrm{d}\omega \\
&\qquad + \int\limits_{\Omega_{t_0,t_1}} c(t) \max_{\substack{1\leqslant\beta\leqslant q\\ 1\leqslant\alpha\leqslant n\\ 1\leqslant j\leqslant m}} \left\{G_j^+\left(\pi_{\widehat{xu}}\right), \left|H_\beta^\alpha\left(\pi_{\widehat{xu}}\right) - \frac{\partial x^\alpha}{\partial t^\beta}\right|\right\}. \qquad (45)
\end{aligned}$$

By the definition of the unconstrained penalized multitime variational problem (IVMCP$_\infty(c)$) with the minimax exact interval-valued penalty function, (45) implies that the inequality

$$P_\infty\left(\left(\widehat{x}(\cdot), \widehat{u}(\cdot)\right), c(\cdot)\right) \prec_{LU} P_\infty\left(\left(\overline{x}(\cdot), \overline{u}(\cdot)\right), c(\cdot)\right)$$

holds, contradicting the assumption that $(\overline{x}(\cdot), \overline{u}(\cdot))$ is an LU-optimal solution in the unconstrained penalized multitime variational problem (IVMCP$_\infty(c)$) with the minimax exact interval-valued penalty function for each $c(\cdot) \geqslant \sum_{j=1}^{m} \widehat{\mu}_j(\cdot) + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{q} \left|\widehat{\vartheta}_\beta^\alpha(\cdot)\right|$. Thus, we conclude by (36) that $(\overline{x}(\cdot), \overline{u}(\cdot))$ is also a LU-minimizer in (IVMCP). This completes the proof of this theorem.

4. Illustrative example

Now, we give the example of an invex interval-valued multitime control problem with first-order PDE constraints to illustrate the results established in the paper.

Example 1. Let $\mathcal{T} = \mathbb{R}^2$, $\mathcal{M} = C = \mathbb{R}$ and Ω_{t_0,t_1} be a rectangle (in particular cases, a square) fixed by the diagonal opposite points $t_0 = \left(t_0^1, t_0^2\right)$ and $t_1 = \left(t_1^1, t_1^2\right)$ in $\mathcal{T}$. Then, we consider the following interval-valued multitime control problem with first-order PDE constraints defined by:

$$\min_{(x(\cdot),u(\cdot))} \left[\int_{\Omega_{0,2}} \ln\left(u^2(t)+1\right) \mathrm{d}t^1 \mathrm{d}t^2, \int_{\Omega_{0,2}} \left(\ln\left(u^2(t)+1\right)+1\right) \mathrm{d}t^1 \mathrm{d}t^2 \right]$$

$$\text{s.t. } \frac{\partial x}{\partial t^1}(t) = \frac{\partial x}{\partial t^2}(t) = -2\arctan u(t), \quad t = \left(t^1, t^2\right) \in \Omega_{0,2}, \quad \text{(IVMCP1)}$$

$$x^2(t) - 49 \leqslant 0, \quad t = \left(t^1, t^2\right) \in \Omega_{0,2},$$

$$x^2(t) - 64 \leqslant 0, \quad t = \left(t^1, t^2\right) \in \Omega_{0,2},$$

$$x(0) = x(0,0) = 0, \quad x(2) = x(2,2) = 2\pi,$$

where $t \in \Omega_{t_0,t_1} = \Omega_{0,2} = [0,\ 2]^2$ is a square fixed by the diagonally opposite points $x_0 = x(t_0) = x\left(t_0^1, t_0^2\right) = x(0,0) = 0$ and $x_1 = x(t_1) = x\left(t_1^1, t_1^2\right) = x(2,2) = 2\pi$ in R^2. We assume that we are only interested in affine functions. From the formulation of (IVMCP1), it follows that

$$f\colon \Omega_{0,2} \times \mathbb{R} \times \mathbb{R} \to I(\mathbb{R}),\ G_j\colon \Omega_{0,2} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ j = 1,2,$$

$$H_\beta\colon \Omega_{0,2} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \beta = 1,2,$$

where

$$f(\pi_{xu}) = \left[f^L(\pi_{xu}), f^U(\pi_{xu})\right] = \left[\ln\left(u^2(t)+1\right), \ln\left(u^2(t)+1\right)+1\right],$$

$H_1(\pi_{xu}) = H_2(\pi_{xu}) = -2\arctan u(t)$, $G_1(\pi_{xu}) = x^2(t) - 49$, $G_2(\pi_{xu}) = x^2(t) - 64$ are continuously differentiable functions. The closeness condition associated with $H_\beta, \beta = 1,2$, implies that $\frac{\partial x}{\partial t^1}(t) = \frac{\partial x}{\partial t^2}(t)$. By the necessary optimality conditions (6)–(9) (see Theorem 1), it follows that if $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \Gamma\left(\Omega_{0,2}\right)$ is an LU-optimal solution in (IVMCP1). Then, there exist $\overline{\lambda}^L \in \mathbb{R}$, $\overline{\lambda}^U \in \mathbb{R}$ and piecewise smooth functions $\overline{\mu}_1(t)\colon \Omega_{0,2} \to \mathbb{R}_+$, $\overline{\mu}_2(t)\colon \Omega_{0,2} \to \mathbb{R}_+$, $\overline{\vartheta}(t) = \left(\overline{\vartheta}_1(t), \overline{\vartheta}_2(t)\right) : \Omega_{0,2} \to \mathbb{R}^2$ for $t = \left(t^1, t^2\right)$ satisfying the following conditions:

$$2\overline{\mu}_1(t)\overline{x}(t) + 2\overline{\mu}_2(t)\overline{x}(t) + \frac{\partial \overline{\vartheta}_1(t)}{\partial t^1} + \frac{\partial \overline{\vartheta}_2(t)}{\partial t^2} = 0,$$

$$\frac{2\overline{\lambda}^L \overline{u}(t)}{\overline{u}^2(t)+1} + \frac{2\overline{\lambda}^U \overline{u}(t)}{\overline{u}^2(t)+1} - \frac{2\overline{\vartheta}_1(t)}{\overline{u}^2(t)+1} - \frac{2\overline{\vartheta}_1(t)}{\overline{u}^2(t)+1} = 0,$$

$$\overline{\mu}_1(t)\left(\overline{x}^2(t)-49\right)=0,$$

$$\overline{\mu}_2(t)\left(\overline{x}^2(t)-64\right)=0,$$

$$\overline{\lambda}^L>0,\ \ \overline{\lambda}^U>0,\ \ \overline{\lambda}^L+\overline{\lambda}^U=1,\ \ \overline{\mu}_1(t)\geqslant 0,\ \ \overline{\mu}_2(t)\geqslant 0$$

for all $t\in\Omega_{0,2}$, except at discontinuities.

Now, we consider the case when $\overline{\vartheta}_1(\cdot)$ and $\overline{\vartheta}_2(\cdot)$ are constant functions. Therefore, we set $\overline{\vartheta}_1(\cdot)=\delta_1$, $\overline{\vartheta}_2(\cdot)=\delta_2$, where $\delta_1,\ \delta_2\in\mathbb{R}$. From third and fourth necessary optimality conditions, it follows that $\overline{x}(t)=\pm 7$ or $\overline{\mu}_1(t)=0$ and $\overline{x}(t)=\pm 8, \overline{\mu}_2(t)=0, t\in int\Omega_{0,2}$, respectively. Note that $\overline{x}(t)=\pm 7$ and $\overline{x}(t)=\pm 8$, $t\in int\Omega_{0,2}$, are not satisfied the boundary condition $x(2)=x(2,2)=2\pi$. Hence, it is not possible that there exist $t=\left(t^1,t^2\right)\in int\Omega_{0,2}$ such that $(\overline{\mu}_1(t),\overline{\mu}_2(t))>0$. Hence, $(\overline{\mu}_1(t),\overline{\mu}_2(t))=(0,0)$ and, therefore, the first necessary optimality conditions reduces to

$$\frac{\partial\overline{\vartheta}_1(t)}{\partial t^1}+\frac{\partial\overline{\vartheta}_2(t)}{\partial t^2}=0.$$

The aforesaid equation admits the following solution $\overline{\vartheta}(t)=\left(\overline{\vartheta}_1(t),\overline{\vartheta}_2(t)\right)=\left(\delta^1,\delta^2\right)$, where $\delta^1,\delta^2\in\mathbb{R}$. Thus, the second necessary optimality conditions gives

$$\left(\overline{\lambda}^L+\overline{\lambda}^U\right)\overline{u}(t)=\delta^1+\delta^2,\ \forall t\in\Omega_{0,2}\,.$$

Let us set $\overline{\lambda}^L+\overline{\lambda}^U=1$. Thus,

$$\overline{u}(t)=\delta^1+\delta^2,\ \ \forall t\in\Omega_{0,2}\,. \tag{46}$$

Hence, by (46), the first constraint of (IVMCP1) yields

$$\overline{x}(t)=-2\left(t^1+t^2\right)\arctan\left(\delta^1+\delta^2\right)+\delta^3,\ \ \forall t\in\Omega_{0,2}\,. \tag{47}$$

Then, the boundary condition $x\,(0,0)=0$ implies that $\delta^3=0$. Moreover, the boundary condition $x(2,2)=2\pi$ implies that $\delta^1+\delta^2=-1$. Thus, we obtain that

$$\overline{x}(t)=\frac{\pi}{2}\left(t^1+t^2\right),\quad \overline{u}(t)=-1,\ \ \forall t\in\Omega_{0,2}\,.$$

Moreover, the necessary optimality conditions are fulfilled at $(\overline{x}(t),\overline{u}(t))$ with given above with Lagrange multipliers

$$\overline{\lambda}^L>0,\ \ \overline{\lambda}^U>0,\ \ \overline{\lambda}^L+\overline{\lambda}^U=1,\ \ (\overline{\mu}_1(t),\overline{\mu}_2(t))=(0,0),\ \ \overline{\vartheta}_1(t)+\overline{\vartheta}_2(t)=1.$$

Now, we show that $(\overline{x}(t), \overline{u}(t)) = \left(\frac{\pi}{2}\left(t^1+t^2\right), -1\right)$ at $t_1 = 0$, $t_2 = 0$ is an LU-optimal solution in (IVMCP1). Note that we have

$$F\left(\overline{x}(\cdot), \overline{u}(\cdot)\right) = \left[\int\limits_{\Omega_{0,2}} \ln\left(\overline{u}^2(t)+1\right) \mathrm{d}t^1 \mathrm{d}t^2, \int\limits_{\Omega_{0,2}} \left(\ln\left(\overline{u}^2(t)+1\right)+1\right) \mathrm{d}t^1 \mathrm{d}t^2\right]$$
$$= \left[\ln 2, 1+\ln 2\right].$$

Moreover, by the PDE constraint $\frac{\partial x}{\partial t^1}(t) = \frac{\partial x}{\partial t^2}(t)$, one has $x(t) = \psi_1\left(t_1+t_2\right)+\psi_2$, where ψ_1, $\psi_2 \in \mathbb{R}$. Hence, by the boundary conditions, we have that $x(t) = \frac{\pi}{2}\left(t_1+t_2\right)$. Then, by Definition 2, there is no other $(x(\cdot), u(\cdot)) \in \Gamma\left(\Omega_{0,2}\right)$ such that $F\left(x(\cdot), u(\cdot)\right) \prec_{LU} F\left(\overline{x}(\cdot), \overline{u}(\cdot)\right)$ which means that, in fact, $(\overline{x}(\cdot), \overline{u}(\cdot)) = \left(1, \frac{3}{2}\right)$ is an LU-optimal solution in (IVMCP1).

Now, we consider the case when at least one of $\overline{\vartheta}_1(\cdot)$ or $\overline{\vartheta}_2(\cdot)$ is not a constant function. For, example, we consider the case when $\overline{\vartheta}_1(t) = t_1$ and $\overline{\vartheta}_2(t) = 0$. By direct computation, we obtain that $\overline{x}(t) = \frac{\pi}{2}\left(t^1+t^2\right)+\ln\left(t_1^2+1\right)+\delta^4$. Then, by the first boundary condition, it follows that $\delta^4 = 0$. Further, note that the second boundary condition is not satisfied in this case since $\overline{x}(2) = 2\pi+\ln 5 \neq 2\pi$. We have also the same in the case when $\overline{\vartheta}_1(t) = 0$ and $\overline{\vartheta}_2(t) = t_2$ and in the case when $\overline{\vartheta}_1(t) = t_1$ and $\overline{\vartheta}_2(t) = t_2$. Then, there do not exist optimal solutions also in the two foregoing cases. Hence, there is no solution in the case when at least one of $\overline{\vartheta}_1(\cdot)$ or $\overline{\vartheta}_2(\cdot)$ is not a constant function.

We now use the minimax exact penalty function method for solving the interval-valued multitime control problem (IVMCP1) considered in this example. Therefore, for the original interval-valued multitime control problem (IVMCP1) with first-order PDE constraints, we construct the unconstrained interval-valued multitime control problem (IVMCP1$_\infty(c)$) with the minimax exact penalty function as follows

$$P_\infty\left((x(t), u(t)), c(t)\right) = \left[\int\limits_{\Omega_{t_0,t_1}} \left(\ln\left(u^2(t)+1\right)+c(t)p\left(t, x(t), u(t)\right)\right) \mathrm{d}t^1 \mathrm{d}t^2,\right.$$

$$\left.\int\limits_{\Omega_{t_0,t_1}} \left(\ln\left(u^2(t)+1\right)+1+c(t)p\left(t, x(t), u(t)\right)\right) \mathrm{d}t^1 \mathrm{d}t^2\right] \to \min, \qquad (\text{IVMCP1}_\infty(c))$$

where

$$p(t,x(t),u(t)) = \max\Big\{\max\left\{0,x^2(t)-49\right\},\ \max\left\{0,x^2(t)-64\right\},$$
$$2\arctan u(t) + \frac{\partial x}{\partial t^\beta}(t),\ \beta=1,2\Big\}.$$

It can be shown, by Definitions 5 and 4, that all functionals involved in (IVMCP1) are invex at $(\overline{x}(t),\overline{u}(t))$ on $\mathcal{X}\times\mathcal{U}$ with respect to $\eta\left(t,x(t),u(t),\overline{x}(t),\overline{u}(t)\right) = x(t)-\overline{x}(t)$ and $\xi\left(t,x(t),u(t),\overline{x}(t),\overline{u}(t)\right) = \arctan u(t) - \arctan\overline{u}(t)$. Since all the hypotheses of Theorem 2 are fulfilled, therefore, $(\overline{x}(t),\overline{u}(t)) = \left(\frac{\pi}{2}\left(t^1+t^2\right),-1\right)$ at $t_1 = 0$, $t_2 = 0$, being an LU-optimal solution in (IVMCP1), is also an LU-optimal solution in the unconstrained interval-valued multitime control problem (IVMCP1$_\infty(c)$) with the minimax exact penalty function for any penalty parameter $c(t) \geqslant \sum_{j=1}^{m}\overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}\left|\overline{\vartheta}_\alpha^\beta(\cdot)\right| = 1$. Conversely, since $(\overline{x}(t),\overline{u}(t)) = \left(\frac{\pi}{2}\left(t^1+t^2\right),-1\right)$ at $t_1 = 0$, $t_2 = 0$ is an LU-optimal solution in (IVMCP1$_\infty(c)$) for any penalty parameter $c(t) \geqslant \sum_{j=1}^{m}\overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{q}\left|\overline{\vartheta}_\alpha^\beta(\cdot)\right| = 1$ and all the hypotheses of Theorem 3 are also fulfilled, therefore, $(\overline{x}(t),\overline{u}(t))$ is also an LU-optimal solution in the considered interval-valued multitime control problem (IVMCP1). Then, there is the equivalence between these aforesaid interval-valued multitime control problems.

Now, we illustrate the case in which there is no any threshold $\overline{c}(\cdot)$ of a penalty parameter such that, for any penalty parameter exceeding $c(\cdot)$, there is the equivalence between the original multitime control problem with first-order PDE constraints and its associated penalized control problem constructed in the minimax exact penalty function method in the sense discussed in the paper. Namely, we show the importance of invexity hypotheses in proving the aforesaid equivalence.

Example 2. Let $q = 2$, $n = 1$ and Ω_{t_0,t_1} be a rectangle (in particular cases, a square) fixed by the diagonally opposite points $t_0 = \left(t_0^1,t_0^2\right)$ and $t_1 = \left(t_1^1,t_1^2\right)$ in

$\mathbb{R}^2$. Let us consider the following interval-valued multitime control problem:

$$\min_{(x(\cdot),u(\cdot))} \left[\int_{\Omega_{0,1}} \left(1+u^3(t)\right) \mathrm{d}t^1 \mathrm{d}t^2, \int_{\Omega_{0,2}} \left(2+u^3(t)\right) \mathrm{d}t^1 \mathrm{d}t^2 \right]$$
$$\text{s.t. } \frac{\partial x}{\partial t^1}(t) = \frac{\partial x}{\partial t^2}(t) = -u(t), \tag{IVMCP2}$$
$$x^2(t) - 9 \leqslant 0,$$
$$t = \left(t^1, t^2\right) \in \Omega_{0,1}, \quad x(0) = x(0,0) = 0, \quad x(1) = x(1,1) = 0.$$

We use the minimax exact penalty function method for solving (IVMCP2). Then, we construct its associated control penalized problem (IVMCP2) as follows

$$P_\infty((x(t),u(t)), c(t)) = \left[\int_{\Omega_{0,1}} \left(1+u^3(t)+c(t)p\left(t,x(t),u(t)\right)\right) \mathrm{d}t^1 \mathrm{d}t^2, \right.$$
$$\left. \int_{\Omega_{0,1}} \left(2+u^3(t)+c(t)p\left(t,x(t),u(t)\right)\right) \mathrm{d}t^1 \mathrm{d}t^2 \right], \tag{IVMCP2$_\infty(c)$}$$

where

$$p\left(t,x(t),u(t)\right) = \max\left\{\max\left\{0, x^2(t)-9\right\}, u(t) + \frac{\partial x}{\partial t^\beta}(t), \ \beta = 1,2\right\}.$$

It can be shown that $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \Gamma\left(\Omega_{t_0,t_1}\right)$, where $\overline{x}(t) = 0$, $\overline{u}(t) = 0$, $\forall t \in \Omega_{0,1}$, is an LU-optimal solution in (IVMCP2) at which the necessary optimality conditions are fulfilled with Lagrange multipliers $\overline{\lambda}^L > 0$, $\overline{\lambda}^U > 0$, $\overline{\lambda}^L + \overline{\lambda}^U = 1$, $\overline{\mu}_1(t) = 0$, $\forall t \in \Omega_{0,1}$, $\overline{\vartheta}(t) = \left(\overline{\vartheta}^1(t), \overline{\vartheta}^2(t)\right) = (\delta_1, \delta_2)$, where $\delta_1, \delta_2 \in \mathbb{R}$ are constant such that $\delta_1 + \delta_2 = 0$. However, note that $(\overline{x}(\cdot), \overline{u}(\cdot))$ is not an LU-optimal solution in (IVMCP2$_\infty(c)$) for any penalty parameter $c(\cdot) > 0$. This follows from the fact that the downward order of growth of the objective function exceeds the upward of growth of the penalty term in the objective function of (IVMCP2$_\infty(c)$) when moving from $\overline{u}(\cdot)$ towards smaller values. This is a consequence of the fact that not all functionals constituting (IVMCP2) are invex at $(\overline{x}(\cdot), \overline{u}(\cdot))$ on $\Gamma\left(\Omega_{0,1}\right)$.

Remark 2. *As it follows even from Example 2, invexity assumption of the functionals involved in (IVMCP2) is important in proving the equivalence between the* *sets of LU-optimal solutions in an interval-valued multitime control problem with first-order PDE constraints and its associated interval-valued penalized control problem constructed in the minimax exact penalty function method. If the invexity assumption is violated, then it is not possible to find LU-optimal solutions in the original interval-valued multitime control problem (IVMCP) by looking for LU-optimal solutions in its associated interval-valued multitime control penalized problem (IVMCP$_\infty(c)$) (which is constructed in the minimax exact penalty function method) for any penalty parameter c. Hence, the fact that there is no the threshold of the penalty parameter in the aforesaid case is important from the practical point of view.*

5. The procedure of the minimax exact penalty function method

Based on the considerations in the paper, we construct the procedure of the minimax exact penalty function method in the case when it is used for solving the original interval-valued multitime control problem (IVMCP) with first-order PDE constraints. We call it IVMEPFM procedure. The primary goal of the presented algorithm is to demonstrate the specific actions required to solve an interval-valued control problem with first-order PDE constraints similar to those ones mentioned in the paper. The aim of this procedure is to find an LU-optimal solution in (IVMCP).

Procedure IVMEPFM

► **Input**

Functions constituting the original interval-valued multitime control problem (IVMCP) with first-order PDE constraints and a set of some self data:

- interval-valued objective function

$$F\left(x(\cdot),u(\cdot)\right) := \int\limits_{\Omega_{t_0,t_1}} \left[f^L\left(t,x(t),u(t)\right), f^U\left(t,x(t),u(t)\right)\right] \mathrm{d}\omega,$$

- set of constraints:
  - $G_j\left(t,x(t),u(t)\right) \leqslant 0,\ j=1,\ldots,m,$
  - $\dfrac{\partial x^\alpha}{\partial t^\beta} = H^\alpha_\beta\left(t,x(t),u(t)\right),\ \alpha=1,\ldots,n,\ \beta=1,\ldots,q,$
  - $x\left(t_0\right)=x_0,\ x\left(t_1\right)=x_1.$
- set of self data:

- (IVMCP) is solvable,
- $H_\beta\left(t, x(\cdot), u(\cdot)\right), \beta = 1, \ldots, q$, satisfy the closeness condition,
- $x(\cdot)$ is an affine state function,
- all the functions constituting (IVMCP) satisfy appropriate invexity hypotheses,
- $\overline{\lambda}^L \in \mathbb{R}_+$, $\overline{\lambda}^U \in \mathbb{R}_+$, piecewise smooth functions $\overline{\mu}\colon \Omega_{t_0,t_1} \to \mathbb{R}^m_+$, $\overline{\xi}\colon \Omega_{t_0,t_1} \to \mathbb{R}^{qn}$,
- the penalty parameter $c(\cdot)$ satisfies $c(\cdot) \geqslant \sum_{j=1}^{m} \overline{\mu}_j(\cdot) + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{q} \left|\overline{\xi}^\alpha_\beta(\cdot)\right|$.

▶ **Begin**

- Selection stage: select $(\overline{x}(\cdot), \overline{u}(\cdot)) \in \Omega_{t_0,t_1}$

if the KKT conditions are fulfilled at $(\overline{x}(\cdot), \overline{u}(\cdot))$ with $\overline{\lambda}^L \in \mathbb{R}_+$, $\overline{\lambda}^U \in \mathbb{R}_+$, $\overline{\mu}(\cdot)$, $\overline{\xi}(\cdot)$ **then**

go to the next stage

else

stop

end if;

- Screening stage: choosing $\overline{\lambda}^L \in \mathbb{R}_+$, $\overline{\lambda}^U \in \mathbb{R}_+$, $\overline{\mu}(\cdot)$ and $\overline{\xi}(\cdot)$

if they satisfy set of self data (that is, hypotheses a)-c) of Theorem 2 are fulfilled) **then**

go to the next stage

else

stop

end if;

- Final stage: having $(\overline{x}(\cdot), \overline{u}(\cdot))$, $\overline{\mu}(\cdot)$ and $\overline{\xi}(\cdot)$ from the previous stages

if $\nexists_{(x(\cdot),u(\cdot))\in \mathcal{X}\times\mathcal{U}}\ P_\infty((x(\cdot), u(\cdot)), c(\cdot)) \prec_{LU} P_\infty((\overline{x}(\cdot), \overline{u}(\cdot)), c(\cdot))\ \forall c(\cdot)$ satisfying (13) **then**

$(\overline{x}(\cdot), \overline{u}(\cdot))$ is an LU-optimal solution in (IVMCP$_\infty(c)$)
and, therefore, it is a LU-minimizer in (IVMCP).

else

stop

end if;

► **End.**

---

6. Conclusions

In the paper, the interval-valued multitime control problem with first-order PDE constraints has been investigated. Namely, one of the methods, as it is the minimax exact penalty function method, for solving the aforesaid interval-valued extremum problem has been investigated. Then, the most important property from a practical point of view of any exact penalty function method has been generalized to the case when the aforesaid exact penalty function method has been used for solving the considered interval-valued multitime control problem with first-order PDE constraints. Thus, this property has been analyzed in the context where the equivalence has been established between LU-optimal solutions in the original interval-valued multitime control problem with first-order PDE constraints and its associated unconstrained penalized multitime variational problem with the interval-valued minimax exact penalty function which has been constructed in the aforesaid exact penalty function method. The aforesaid result has been established for sufficiently large penalty parameters and under appropriate invexity hypotheses. Thus, the minimax exact penalty function method turned out to be an effective tool in finding LU-optimal solutions of the considered interval-valued multitime control problem with first-order PDE constraints by looking for LU-optimal solutions of its associated penalized interval-valued multitime variational problem constructed in the used minimax exact penalty function method which is an unconstrained control problem. Finally, the procedure of using the minimax exact penalty function method for finding LU-optimal solutions of the analyzed interval-valued extremum problem has been presented.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to use the interval-valued minimax exact penalty function method for other classes of interval-valued multitime control problem with first-order PDE constraints. We shall investigate these questions in subsequent papers.

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