

Design of a positive observer-based controller for positive linear systems

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In this paper, we investigate the design of a positive functional observer-based controller for positive linear systems subject to unknown inputs. We propose a new approach to numerically compute the controller, which is obtained via a function of the state to be estimated with the same order as the controller. The positive observer-based controller, is obtained into two steps. First, we search for a positive state feedback gain for the design of a control law such that the closed-loop is positive and asymptotically stable. This synthesis of state-feedback gain is solved using Linear Programming (LP) technique. Then, we search in a second step for a positive functional observer-based controller permitting to reconstruct this control law, and so to estimate only a functional of the state useful for control purposes. The observer is positive, that is, it ensures the nonnegativity of the estimated states. The proposed procedure is based on the positivity of an augmented system composed of dynamics of both considered system and proposed observer-based controller and also, on the unbiasedness of the estimation error by the resolution of Sylvester equation. Then existence conditions of such observer-based controller are formulated in terms of linear programming (LP) problem, where we use the Perron-Frobenius theorem applied to Metzler matrices. An algorithm that summarizes the different steps of the designed positive controller for positive linear systems is given. Finally, a numerical example and simulation results are given to illustrate the effectiveness of the proposed method.

Key words: positive systems, positive functional observers, observer-based controller, unknown inputs, linear systems, linear programming (LP) problem

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Received 7 September 2025. Accepted 7 July 2026.

1. Introduction

In many physical systems, variables are constrained to be positive. For, example, absolute temperature and concentration of substances in chemical processes. These examples belongs to an important class of dynamical systems whose states are nonnegative for any nonnegative initial condition and any nonnegative input. In the literature, such systems are referred to be positive (see [5, 6, 15, 23, 31]). In view of these widespread applications, it is very important and necessary to investigate the study and synthesis problems for positive linear systems (see [14, 19]). Note that, positive systems differ from linear standard systems by the existence of positivity constraints. In fact, for example, if a system is controllable, the poles of the system can be placed arbitrarily, whereas for positive one this feature may not be true owing the positivity constraints on systems matrices.

On the other hand, in practice, many control processes require the availability of the components of the system state vector for the purpose of monitoring for example. This problem, has motivated a great deal of work for observers design for linear systems, (see [7, 24, 30]) when the states are not measurable or not available. Mainly, problem of functional observer design is equivalent to find an observer, that estimates a linear combination of the states of a system using the input and output measurements. Such estimator has the same order as the linear combination to be estimated. Note that, numerous studies for non positive classical systems the objective is only to minimize the estimation error (make the estimation error converge to zero), (see [7, 8, 11]). For positive systems, in addition to minimizing this estimation error, positive observers must also guarantee the nonnegativity of the state estimates, (see [9, 12, 13]). This makes the positive observer design significantly more challenging, (see [3], [22, 26]). For positive systems with unknown inputs (see [13, 28]) a design method via LMIs is provided. In [21] and [29], unknown input observers for discrete-time positive systems, state and unknown input observers for positive systems were reported.

Notice that, in order to ensure stability and optimal performance, it is imperative to consider the positivity constraint when controlling positive systems. Failure to do so may lead the mathematical model of these systems to venture into infeasible regions that are beyond the reach of the real system. Consequently, this can result in a loss of stability or performance when implementing the controller on real plant.

Moreover the observer-based control is usually used when all the states of a system are not accessible (see [18, 32, 33]). It is interesting to recall that feedback principle is an important concept in control theory especially for positive systems. It is based on algebraic approaches in [4, 10], on quadratic programming problem in [20], on Linear Matrix Inequality (LMI) in [17], on analytical methods in [16]

and on linear programming (LP) in [1]. Indeed many different control strategies are based on the assumption that all internal states of the control object are available for feedback. In most cases, however, only a few part of the states or some functions of the states can be measured. Then the need for techniques which make it possible not only to estimate states, but also to derive control laws that guarantee stability and positivity when using the estimated states instead of the true ones, is crucial.

Despite the recent advances, to the best of our knowledge, little attention has been paid for the positive functional observer-based controller design for positive linear systems subject to unknown inputs, which motivates the present work. This is an important problem finding its way into multiple engineering applications such as in fault detection. It is also an important research topic since these kind of controllers are very important in practice as they possess real physical meaning. These are some motivations of the present work.

Notice that, in this paper we means by unknown inputs, inputs that are totally unknown not even the borders of their energy, which is of practical interest. As far as the authors knowledge, the proposed approach is original.

Motivated by these facts, we consider in this paper, a new problem of designing positive functional observer-based controller for positive linear systems subject to unknown inputs. We propose a simple approach to solve numerically the computation of the controller, where the order of the observer-based controller is equal to the state feedback that is find to fulfill the control purpose. The positive observer-based controller is obtained into two steps. First, we search for a positive state feedback gain for the design of a control law such that the closed-loop is asymptotically stable and positive. This synthesis of positive state-feedback gain is solved using Linear Programming (LP) technique. Second, we search for a positive observer-based controller permitting to reconstruct this control law, and so to estimate only a function of the state useful for control purposes. The proposed procedure is based on the positivity of an augmented system composed of dynamics of both considered system and proposed observer-based controller and also, on the unbiasedness (see [7]) of the estimation error by the resolution of Sylvester equation. Then existence conditions of such observer-based controller are formulated also, in terms of linear programming (LP) problem, where we use the Perron-Frobenius theorem applied to Metzler matrices [2]. An algorithm that summarizes the different steps of the proposed controller based on positive functional observers for positive linear systems design is given. Finally, numerical example and simulation results are given to illustrate the effectiveness of the proposed design method.

Recall that the most interest of our approach, is the use of a functional observer (not a full order one) which estimates the desired control law, without estimating

all the state of the system contrary to full order observer-based control approach. The interest is the reduction of the order of the controller since the designed functional observer is of the same order than the functional to be estimated, i.e. the dimension of the control input $u(t)$.

Notations: We shall use throughout the paper the following notations:

- Let $\mathfrak{R}$ be the set of real numbers; $\mathfrak{R}_+^n$ denotes the nonnegative orthant of the n -dimensional real space $\mathfrak{R}^n$ and $\mathfrak{R}^{m \times n}$ is the set of $m \times n$ matrices for which all entries belong to $\mathfrak{R}$.
- For a matrix $A \in \mathfrak{R}^{n \times m}$, a_{ij} denotes the element located at the i th row ($i \leq n$) and j th column ($j \leq m$).
- A matrix A is said to be nonnegative, denoted by $A \succcurlyeq 0$, if $\forall(i, j), a_{ij} \geq 0$. It is said to be positive, if $\forall(i, j), a_{ij} \geq 0, \exists(i, j), a_{ij} > 0$. Note that definitions of nonnegative and positive matrices are equivalent, except when a nonnegative matrix is identically zero which is the degenerate case and is of no interest. So, we consider that these two definitions are equivalent in general cases.
- A matrix A is said to be negative, denoted by $A \prec 0$, if $\forall(i, j), a_{ij} \leq 0$.
- $A > 0$ (respectively, $A < 0$) means that the matrix A is positive definite (respectively, negative definite), $A \geq 0$ (respectively, $A \leq 0$) means that the matrix A is positive semidefinite (respectively, negative semidefinite).
- For a real matrix A , A^T denotes the transpose.
- A^- denotes any generalized inverse of matrix A , i.e. it verifies $AA^-A = A$.
- $\lambda_i(A)$ designs the i th eigenvalue of matrix A .
- A is Hurwitz matrix if all real parts of its eigenvalues are in the left half plane.
- $\text{diag}(v)$ denotes the diagonal matrix formed from the vector v .
- $\text{Ones}(n)$ denotes an $n \times 1$ vector of ones $:= (1, \dots, 1)^T$.
- I and 0 are the identity matrix and the zero matrix of appropriate dimensions.
- $v \succ 0$ denotes a vector v such that for all its coordinates it holds, $\forall i, v_i \geq 0$.

2. Preliminaries

In this section, we recall some results that will be essential in the proposed design. For that, let us consider the following continuous linear systems of the form:

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t), \quad (1a)$$

$$y(t) = \tilde{C}x(t) + \tilde{D}u(t), \quad (1b)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^m$ are the system state, (exogenous) input and the external output respectively. $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are system matrices with compatible dimensions.

By following definitions, we recall basic results on positive linear standard systems.

Definition 1. ([28]) *The system in (1) is called internally positive (Metzlerian system) if for every initial condition, $x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^p$, $t \geq 0$ we have $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^m \forall t \geq 0$.*

Definition 2. ([28]) *A linear system (1) is said to be positive if its state and output are both nonnegative ($x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^m \forall t \geq 0$) for any nonnegative input and nonnegative initial state.*

Definition 3. ([25]) *A square real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative, i.e. $m_{ij} \geq 0$, $i \neq j$.*

Then, we present following lemmas which will be used in the sequel of the paper.

Lemma 1. ([28]) *System (1) is positive if and only if $\tilde{A}$ is a Metzler matrix and $\tilde{B} \in \mathbb{R}_+^{n \times p}$, $\tilde{C} \in \mathbb{R}_+^{m \times n}$, $\tilde{D} \in \mathbb{R}_+^{m \times p}$ are nonnegative matrices: ($\tilde{B} \geq 0$, $\tilde{C} \geq 0$ and $\tilde{D} \geq 0$).*

Lemma 2. ([2]) *The Perron-Frobenius theorem applied to matrices $S \in \mathbb{M} \cap \mathbb{H}$, where $\mathbb{M}$ denotes the set of Metzler matrices and $\mathbb{H}$ denotes the set of Hurwitz matrices, states that: for all $S \in \mathbb{M} \cap \mathbb{H}$, there exists a nonnegative vector v i.e. $v \geq 0$, such that $Sv < 0$.*

Lemma 3. ([28]) *Let the system (1) be Metzlerian. Then it is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1. *All eigenvalues of the matrix A have negative real parts; then A is a Hurwitz matrix.*
2. *All coefficients of the characteristic polynomial $\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ are positive, i.e. $a_i > 0 \forall i = 0, 1, \dots, n-1$.*
3. *All leading principal minors of $-A$ are positive.*
4. *The matrix A is nonsingular and $-A^{-1} \geq 0$.*
5. *There exists a positive definite (possibly diagonal) matrix P such that $A^T P + PA < 0$.*
6. *There exists a nonnegative vector v i.e. $v \geq 0$, such that $Av < 0$.*

Lemma 4. A matrix G is a generalized inverse of positive matrix $\tilde{A}$ if it satisfies $\tilde{A}G\tilde{A} = \tilde{A}$.

Let $\tilde{A}^-$ be one generalized inverse of $\tilde{A}$ i.e. $\tilde{A}\tilde{A}^-\tilde{A} = \tilde{A}$, then

$$G = \tilde{A}^- + \tilde{V}(I - \tilde{A}\tilde{A}^-) + (I - \tilde{A}^-\tilde{A})\tilde{W}, \quad (2)$$

where $\tilde{V}$ and $\tilde{W}$ are arbitrary matrices.

In fact it is easy to see that

$$\tilde{A}G\tilde{A} = \tilde{A}\tilde{A}^-\tilde{A} + \tilde{A}\tilde{V}(I - \tilde{A}\tilde{A}^-)\tilde{A} + \tilde{A}(I - \tilde{A}^-\tilde{A})\tilde{W}\tilde{A} = \tilde{A}\tilde{A}^-\tilde{A} = \tilde{A}. \quad (3)$$

Now we are ready to state the main problem considered in this work.

3. Problem statement

Let us consider the following linear multivariable continuous-time system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Dd(t), \quad (4a)$$

$$z(t) = F_1x(t), \quad (4b)$$

$$y(t) = Cx(t), \quad (4c)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the control input, $d(t) \in \mathbb{R}^q$ represents the unknown inputs vector, $y(t) \in \mathbb{R}^m$ is the output vector and $z(t) \in \mathbb{R}^{m_z}$ is the controlled output. A , B , D , F_1 and C are known constant matrices of appropriate dimensions.

Further, it is assumed through the paper that:

Assumption 1.

1) $\text{rank } F_1 = m_z$, $m_z \leq n$.

2) $\text{rank } C = m$, $m \leq n$.

Let us now assume that the considered linear system defined in (4) is positive. From Definition 1 and for every initial condition, $x_0 \in \mathbb{R}_+^n$, all inputs $u(t) \in \mathbb{R}_+^p$, $d(t) \in \mathbb{R}_+^q$ $t \geq 0$ we have $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^m$, $z(t) \in \mathbb{R}_+^{m_z} \forall t \geq 0$. In addition, from Lemma 1 we have that, A is a Metzler matrix and $B \in \mathbb{R}_+^{n \times p}$, $D \in \mathbb{R}_+^{n \times q}$, $F_1 \in \mathbb{R}_+^{m_z \times n}$, $C \in \mathbb{R}_+^{m \times n}$ are nonnegative matrices.

Our aim is to design a positive functional observer-based controller, which has the following Luenberger observer form:

$$\dot{\varphi}(t) = N\varphi(t) + Jy(t) + Hu(t), \quad (5a)$$

$$\hat{u}(t) = \varphi(t) + Ey(t), \quad (5b)$$

where $\varphi(t) \in \mathfrak{R}^p$ (with $p \leq n$) is the observer state, $\hat{u}(t)$ is the output of the observer and matrices N , J , H and E are to be designed.

A block diagram of the proposed observer-based controller is shown in Figure 1.

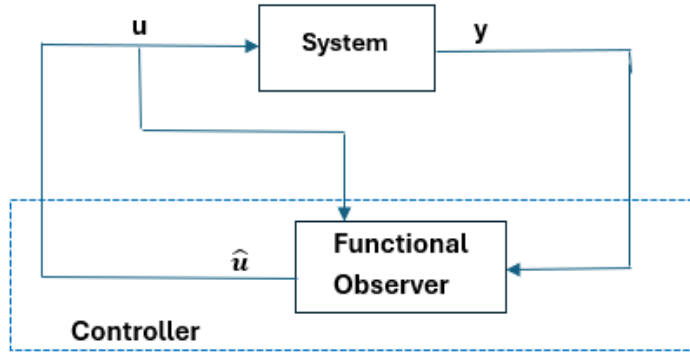


Figure 1: Structure of the observer-based controller

Then, the problem to be investigated can be stated as follows:

Objective 1. *Our main objective is to build for the positive linear system (4) subject to unknown inputs, a positive functional observer-based controller into two steps:*

- i) *First, we search for a positive state feedback gain $K \succ 0$ for the design of a control law $u = Kx$ such that the closed-loop of system (4) is positive and asymptotically stable; $\lim_{t \rightarrow \infty} (u - Kx) = 0$.*
- ii) *Then, we search in a second step for a positive observer-based controller (5), that ensures the nonnegativity of the estimates of nonnegative states, permitting to reconstruct this control law such that the estimation error $e(t) = \hat{u}(t) - Kx(t)$ is stable and converges asymptotically to zero as $t \rightarrow \infty$.*

Notice that the order of this controller is equal to the dimension of the function to be estimated.

Before continuing, we give the following definition:

Definition 4. *The functional observer defined in (5) is called a positive linear functional observer based-controller for system (4) if the output of the observer (5) $\hat{u}(t) \in \mathfrak{R}_+^p$ for all $t \geq 0$ and converges asymptotically to the functional $Kx(t)$ as $t \rightarrow \infty$.*

4. Positive observer-based controller

Here, we propose to solve the positive observer-based control problem into the two steps proposed in Objective 1.

4.1. State-feedback synthesis

Let us assume that the state variable $x(t)$ can be directly measured. We propose in this part to design a positive state feedback control law of the following form:

$$u(t) = Kx(t), \quad (6)$$

such that the closed-loop system given by (4)-(6):

$$\dot{x}(t) = (A + BK)x(t) + Dd(t), \quad (7a)$$

$$z(t) = F_1x(t) \quad (7b)$$

is asymptotically stable and positive, where $K \succ 0$ is the state feedback gain.

Using Lemma 1 and Definition 1, one can state the following lemma:

Lemma 5. *System (7) is positive if and only if $(A + BK)$ is a Metzler matrix and $D \in \mathfrak{R}^{n \times q}$ and $F_1 \in \mathfrak{R}^{m_z \times n}$ are nonnegative matrices: ($D \succ 0$ and $F_1 \succ 0$).*

With regard to previous results, the problem reduces to find conditions satisfying:

- 1) $A + BK$ is a Metzler matrix, with $D \succ 0$ and $F_1 \succ 0$.
- 2) $A + BK$ is a Hurwitz matrix.

So, we are in position to state the main result of this section.

Theorem 1. *Consider the positive linear system (4), where $A = [a_{ij}] \in \mathfrak{R}^{n \times n}$ and $B = [b_{ij}] \in \mathfrak{R}^{n \times p}$ such that $B^T = [b_1^T, \dots, b_n^T]$. Then a controller of the form (6) satisfying that the closed-loop system given by (7) is positive and asymptotically stable exists if there exist a nonnegative vector $v = [v_1, \dots, v_n]^T \in \mathfrak{R}_+^n$, where $V_1 = \text{diag}(v)$ and $Y_1 \in \mathfrak{R}_+^{p \times n} = [y_{11}, \dots, y_{1n}] \succ 0$ such that the following linear programming (LP) problem is feasible:*

$$\begin{cases} v \succ 0, \\ Av + BY_1 \text{ Ones}(n) \prec 0, \\ a_{ij}v_j + b_i y_{1j} \succ 0, \quad 1 \leq i \neq j \leq n. \end{cases} \quad (8)$$

Under above conditions, the state-feedback gain $K \in \mathfrak{R}_+^{p \times n} = [k_1, \dots, k_n] \succ 0$ can be computed by:

$$K = Y_1 V_1^{-1} \quad (9)$$

$$= \begin{bmatrix} y_{1_1} v_1^{-1} & y_{1_2} v_2^{-1} & \dots & y_{1_n} v_n^{-1} \end{bmatrix}. \quad (10)$$

Proof. The closed-loop system given by (7) is:

1. Positive if $A + BK$ is a Metzler matrix, with $D \succ 0$ and $F_1 \succ 0$ (see Lemma (5)). Matrices D and F_1 are nonnegative since the considered initial system (4) is positive. It is easy to see that $A + BK$ is a Metzler matrix implies for $i \neq j$

$$a_{ij} + b_i k_j \succ 0, \quad 1 \leq i \neq j \leq n \quad (11)$$

which coincides with last relation in (8), by defining $k_j = y_{1_j} v_j^{-1}$.

2. Asymptotically stable if $A + BK$ is Hurwitz. Since $A + BK$ is proved that it is Metzler, from Lemma (2) $A + BK$ is Hurwitz if:

$$\begin{cases} v \succ 0, \\ (A + BK)v \prec 0 \end{cases} \quad (12)$$

which is equivalent to the two first relations of inequalities (8), with $v = V_1 \text{Ones}(n)$ and $Y_1 = KV_1$. This completes the proof.

Remark 1. Note that, Theorem 1 presents a simple approach to solve numerically the computation of the state feedback problem. In fact, conditions of (8), that permit to find the desired state feedback gain and so the desired state feedback control law (6), are linear. So, they can be solved as a Linear programming problem.

Once the state feedback gain obtained, recall that all the states are not really accessible generally. Then, we propose a functional observer-based controller to reconstruct the obtained control law.

4.2. Positive observer-based controller existence conditions

To guarantee the positivity of the designed observer-based controller, we propose to compute the following augmented system consisting of the dynamics of (4) and (5). In fact, it can be given by:

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{\hat{u}}(t) \end{pmatrix} &= \begin{pmatrix} A & 0 \\ JC + ECA - NEC & N \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{u}(t) \end{pmatrix} \\ &+ \begin{pmatrix} B \\ H + ECB \end{pmatrix} u(t) + \begin{pmatrix} D \\ ECD \end{pmatrix} d(t). \end{aligned} \quad (13)$$

By applying Lemma 1 on augmented system (13), we can give the following definition:

Definition 5. (5) is a positive observer of system (4) if:

- 1) N is Metzler,
- 2) $JC + ECA - NEC \succ 0$,
- 3) $H + ECB \succ 0$,
- 4) $ECD \succ 0$.

Furthermore, let us compute the estimation error, that is given by:

$$e(t) = \hat{u}(t) - Kx(t) \quad (14a)$$

$$= \varphi(t) + (EC - K)x(t).. \quad (14b)$$

So, its dynamics can be written as

$$\dot{e}(t) = \dot{\varphi}(t) + (EC - K)\dot{x}(t) \quad (15a)$$

$$= Ne(t) + (H + ECB - KB)u(t) + (ECD - KD)d(t) \\ + (NK - NEC + JC + ECA - KA)x(t). \quad (15b)$$

We are now ready to state the second result of the paper, namely the existence conditions of the proposed functional observer-based controller. In fact, the following theorem provides conditions which ensure that the proposed system (5) is a positive linear functional observer of system (4), by providing an output $\hat{u}(t)$ that is always nonnegative and converges asymptotically to the desired functional $Kx(t)$.

Theorem 2. The observer-based controller defined in (5) is an asymptotic positive linear functional observer-based controller of system (4) if and only if the following conditions are satisfied:

- 1) N is Metzler and Hurwitz matrix,
- 2) $JC + ECA - NEC \succ 0$,
- 3) $H + ECB \succ 0$,
- 4) $ECD \succ 0$,
- 5) $NK - NEC + JC + ECA - KA = 0$,
- 6) $H + ECB - KB = 0$,
- 7) $ECD - KD = 0$.

Proof. The first part of condition 1) (N is Metzler matrix) with conditions 2), 3) and 4) are obtained from definition 5. They ensure that the estimate $\hat{u}(t)$, i.e. the output of the proposed observer (5), be nonnegative all the time. In addition, by considering the expression (15) of the estimation error dynamics, one can conclude that it is unbiased (does not depend explicitly on state $x(t)$ and input

$u(t)$) if and only if conditions 5) and 6) are satisfied with the unknown inputs independent observer condition 7). So, the estimation error dynamics $\dot{e}(t) = Ne(t)$ is asymptotically stable if and only if second part of condition 1) (N is Hurwitz) is satisfied. Finally, it is clear that if conditions of the proposed theorem are satisfied then $\hat{u}(t)$ is always nonnegative and it tends to $Kx(t)$ asymptotically ($e(t)$ tends to zero asymptotically for any initial value $e(0)$; $\lim_{t \rightarrow \infty} e(t) = 0$).

From previous developments, we can give the existence conditions of the proposed positive unknown inputs functional observer-based controller:

Lemma 6. *The unknown input observer-based controller, defined in (5) for positive system (4), exists if:*

$$\text{rank}(CD) = \text{rank} \left[\begin{pmatrix} K \\ C \end{pmatrix} D \right] = q. \quad (16)$$

In fact, condition 7) (unknown inputs independent condition) of Theorem 2 is solvable if (16) is satisfied.

Proof. From the general solution of linear matrix equations (see [27]), there exists a solution to $ECD = KD$ if and only if:

$$(KD)(CD)^-(CD) = KD, \quad (17)$$

where $(CD)^-$ is any generalized inverse matrix of (CD) satisfying $(CD)(CD)^-(CD) = (CD)$. From equation (17) we have

$$\text{rank} \left[\begin{pmatrix} KD \\ CD \end{pmatrix} \right] = \text{rank} \begin{pmatrix} \Omega CD \\ CD \end{pmatrix} \quad (18)$$

$$= \text{rank} \begin{pmatrix} I & -\Omega \\ 0 & I \end{pmatrix} \begin{pmatrix} \Omega CD \\ CD \end{pmatrix} \quad (19)$$

$$= \text{rank}(CD), \quad (20)$$

where $\Omega = (KD)(CD)^-$, this proves the lemma.

We are now ready to state the third result of the paper, namely the design procedure of the proposed functional observer. In fact, the following subsection is devoted to find the functional observer-based controller matrices N , J , H and E such that conditions 1)–7) of Theorem 2 are satisfied.

Before continuing, we will give the following remark on the design of positive functional observer with respect to classical functional observer.

Remark 2. *It is worth noting that the functional observer design for positive systems is significantly more difficult than for systems without nonnegativity constraint. The main feature of the proposed method in this paper, is that the design is reduced to a linear programming problem that makes it interesting and easy to manipulate and implement.*

4.3. Positive observer-based controller design

As first step of the design, let us consider conditions 6) and 7) of Theorem 2. One can get functional observer-based controller matrix E from condition 7) of Theorem (2), if rank condition (16) is verified provided that a nonnegative left inverse of CD exists. (Note that from condition 4) of Theorem (2) matrix E must be positive).

Notice that, a parametrization of all positive generalized inverses of a known and positive matrix $\tilde{A}$ is stated by Lemma 4. Therefore in order to guarantee the positivity of the generalized inverse of CD , it is sufficient to solve an LP problem (see (2)) to find matrices $\tilde{V}$ and $\tilde{W}$ since CD is known and positive ($C \in \mathfrak{R}_+^{m \times n}$ and $D \in \mathfrak{R}_+^{n \times q}$).

So, observer-based controller matrix E can be given by

$$E = KD(CD)^- - Z_1(I - (CD)(CD)^-), \quad (21)$$

where Z_1 is an arbitrary matrix that can be, for simplicity assumed to be zero, which gives that

$$E = KD(CD)^-. \quad (22)$$

Then, we can get functional observer-based controller matrix H from condition 6) of Theorem 2

$$H = KB - KD(CD)^-CB. \quad (23)$$

Now, to achieve unbiasedness of the proposed observer-based controller, the following condition 5) of Theorem 2 must hold:

$$NK - NEC + JC + ECA - KA = 0. \quad (24)$$

The equation (24), which has two unknowns, that are observer matrices N and J , can be transformed to

$$\begin{bmatrix} N & J \end{bmatrix} \begin{bmatrix} T \\ C \end{bmatrix} = TA, \quad (25)$$

where $T = K - EC$. (E is given by (22)).

For the resolution of (25), let set

$$\begin{bmatrix} N & J \end{bmatrix} = X, \quad (26)$$

$$\begin{bmatrix} T \\ C \end{bmatrix} = \Sigma, \quad (27)$$

$$TA = \Theta \quad (28)$$

therefore (25) becomes

$$X\Sigma = \Theta. \quad (29)$$

This equation has a solution X if and only if

$$\text{rank} \begin{pmatrix} \Sigma \\ \Theta \end{pmatrix} = \text{rank} \Sigma \quad (30)$$

in this case the general solution for (29), is given by

$$X = \Theta \Sigma^- - Z (I_{p+m} - \Sigma \Sigma^-), \quad (31)$$

where Σ^- is a generalized inverse of matrix Σ given by (27) and $Z \in \mathbb{R}^{p \times (p+m)}$ is an arbitrary matrix, that will be determined in the sequel. Once matrix X is determined, it is easy to give the expressions of matrices N and J .

In fact,

$$N = X \begin{pmatrix} I_p \\ 0_{m \times p} \end{pmatrix} = A_{11} - Z B_{11}, \quad (32)$$

where

$$A_{11} = \Theta \Sigma^- \begin{pmatrix} I_p \\ 0_{m \times p} \end{pmatrix}, \quad (33)$$

$$B_{11} = (I_{p+m} - \Sigma \Sigma^-) \begin{pmatrix} I_p \\ 0_{m \times p} \end{pmatrix} \quad (34)$$

and

$$J = X \begin{pmatrix} 0_{p \times m} \\ I_m \end{pmatrix} = A_{22} - Z B_{22} \quad (35)$$

with

$$A_{22} = \Theta \Sigma^- \begin{pmatrix} 0_{p \times m} \\ I_m \end{pmatrix}, \quad (36)$$

$$B_{22} = (I_{p+m} - \Sigma \Sigma^-) \begin{pmatrix} 0_{p \times m} \\ I_m \end{pmatrix}. \quad (37)$$

Hence functional observer-based controller matrices N and J are determined if and only if the matrix Z is known.

Now, we propose a method to compute this matrix Z such that conditions 1)–4) of Theorem 2 are satisfied; Note that equations (29), (23) and (22) correspond to conditions 5), 6) and 7) of Theorem 2.

For that, let us note that conditions 3) and 4) of Theorem 2 are all time verified due to the nonnegativity of matrices B , C , D , K and $(CD)^-$. In addition, from condition 1) of Theorem 2, to ensure that matrix N be a Metzler matrix, we must have, for a nonnegative vector $\tilde{v} \in \mathbb{R}_+^p$:

$$N V_2 \succ 0 \text{ for } i \neq j, \quad (38)$$

where, we define $V_2 = \text{diag}(\tilde{v})$.

From Lemma 2 since N is Metzler, N is Hurwitz is equivalent to existence of some vector $\tilde{v} \succ 0$ such following inequality holds:

$$N\tilde{v} \prec 0. \quad (39)$$

Note that we can write, $\tilde{v} = \text{diag}(\tilde{v}) \text{ Ones}(p) = V_2 \text{ Ones}(p)$.

As such, we can rewrite condition 1) and condition 2) of Theorem 2 (that are useful for the positivity of the output of the functional observer-based controller $\hat{u}(t)$) in terms of this vector $\tilde{v}$, as follows:

- N is Hurwitz and Metzler matrix, which can also be written as:

$$\begin{cases} \tilde{v} \succ 0, \\ N^T V_2 \text{ Ones}(p) \prec 0, \\ N^T V_2 \succ 0 \text{ for } 1 \leq i \neq j \leq p. \end{cases} \quad (40)$$

- From condition 2) of Theorem 2, we can write the following equivalent condition

$$(JC + ECA - NEC)^T V_2 \succ 0. \quad (41)$$

At this stage, using equivalent conditions (40)–(41), we're ready to state the third result of the paper that permits to reconstruct the control law $u(t) = Kx(t)$ by obtaining remaining observer-based controller matrices N and J such that conditions 1) and 2) of Theorem (2) are satisfied. Note that based on linear programming problem, one can get the gain matrix Z which parameterizes the positive functional observer-based controller matrices N and J (see (32) and (35)).

So, we give the following theorem which permits to ensure the existence of the unknown input functional positive observer-based controller and to obtain the observer's matrices.

Theorem 3. *The positive unknown inputs functional observer-based controller (5) for positive linear system (4) exists, if the following conditions holds:*

1. Rank conditions (16) and (30) are satisfied.
2. The positivity test of one generalized inverse $(CD)^-$ of CD as mentioned by Lemma (4) is satisfied.
3. The following linear programming (LP) problem in variables $V_2 \in \mathbb{R}^{p \times p}$ and $Y_2 \in \mathbb{R}^{(p+m) \times p} = [y_{21}, \dots, y_{2p}]$ for $\tilde{v} = [\tilde{v}_1, \dots, \tilde{v}_p]^T \in \mathbb{R}^{p \times 1}$, where $V_2 = \text{diag}(\tilde{v})$ is feasible:

$$\begin{cases} \tilde{v} \succ 0, \\ A_{11}^T \tilde{v} - B_{11}^T Y_2 \text{ Ones}(p) \prec 0, \\ \tilde{a}_{ij} \tilde{v}_i - \tilde{b}_j^T y_{2i} \succ 0 \text{ for } 1 \leq i \neq j \leq p, \\ (A_{22}C + E_1A - A_{11}E_1)^T V_2 + (B_{11}E_1 - B_{22}C)^T Y_2 \succ 0, \end{cases} \quad (42)$$

where $A_{11} = [\tilde{a}_{ij}] \in \mathfrak{R}^{p \times p}$, $B_{11} = [\tilde{b}_1, \dots, \tilde{b}_p] \in \mathfrak{R}^{(p+m) \times p}$ and $E_1 = EC$.

If this proposed (LP) problem is feasible, variables V_2 and Y_2 are consequently known. Then, the gain Z is given by

$$Z = V_2^{-1} Y_2^T \quad (43)$$

$$= [\tilde{v}_1^{-1} y_{21} \quad \tilde{v}_2^{-1} y_{22} \quad \dots \quad \tilde{v}_p^{-1} y_{2p}]^T. \quad (44)$$

Proof.

1. Let's consider (40), with (32) and using the fact that $A_{11} = [\tilde{a}_{ij}] \in \mathfrak{R}^{p \times p}$, $B_{11} \in \mathfrak{R}^{(p+m) \times p} = [\tilde{b}_1, \dots, \tilde{b}_p]$ and $Y_2 \in \mathfrak{R}^{(p+m) \times p} = [y_{21}, \dots, y_{2p}]$ for $\tilde{v} = [\tilde{v}_1, \dots, \tilde{v}_p]^T \in \mathfrak{R}^{p \times 1}$, where $V_2 = \text{diag}(\tilde{v})$, yields to the first three inequalities of (42) where $Y_2 = Z^T V_2$.
2. Finally, by considering inequality (41), using (32) and (35), the last inequality of (42) holds with $Y_2 = Z^T V_2$ and $E_1 = EC$.

This completes the proof of the proposed theorem.

In the following section, we intend to summarize the different steps that must be achieved to design the proposed positive unknown inputs functional observer-based controller.

5. Summarized algorithm for the design of positive functional observer based controller

- Step 1.** If the Linear Programming problem (LPb) (8) is feasible, get matrices V_1 and Y_1 .
- Step 2.** Compute the state feedback gain K by (9) for the control law design (6).
- Step 3.** Check existence condition of the observer-based controller (16) and rank condition (30).
- Step 4.** Verify the existence of a nonnegative generalized inverse $(CD)^-$ of (CD) , using Lemma 4.
- Step 5.** Get observer-based controller matrices E and H from (22) and (23).
- Step 6.** Compute Σ and Θ from (27) and (28).
- Step 7.** Verify rank condition (30).

Step 8. Compute A_{11} , B_{11} , A_{22} and B_{22} from relations (33), (34), (36) and (37).

Step 9. If the second Linear Programming problem (LPb) (42) is feasible, get matrices V_2 and Y_2 .

Step 10. Compute observer matrix gain Z from (43)-(44).

Step 11. Get observer-based controller matrices N and J from (32) and (35).

Note that, if rank conditions fails and/or the linear programming problem aren't feasible, the controller based-observer does not exist and we much augment the dimension of the observer to hope to have a functional observer. And if the algorithm runs well until the end, then the proposed positive functional observer-based controller description (5) for positive unknown inputs linear system (4) is obtained.

Note that the summarized algorithm is illustrated in the form of a block diagram in Figure 2.

The following section is devoted to demonstrate the effectiveness of the proposed approach on a numerical example.

6. Numerical results

Consider the system (4) presented in section 3, where

$$A = \begin{bmatrix} -9 & 0 & 0.5 \\ 0.2 & -8 & 1 \\ 0.3 & 1.3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.6 & 0.5 \\ 0.1 & 0.8 \\ 2 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0.1 \\ 1 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 2 & 0.1 \end{bmatrix}.$$

With the computational approach summarized in section 5, which is based on LP problem, we have obtained the following results:

6.1. Positive state-feedback gain synthesis

The proposed LP problem (8) is feasible. One such feasible solution provides:

$$1. \quad V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 7 \end{bmatrix} \quad \text{and} \quad Y_1 = \begin{bmatrix} 0.0208 & 0.0584 & 0.0378 \\ 1.7150 & 1.6554 & 0.8825 \end{bmatrix}.$$

$$2. \quad \text{Then, the state feedback gain } K = \begin{bmatrix} 0.0208 & 0.0058 & 0.0054 \\ 1.7150 & 0.1655 & 0.1261 \end{bmatrix}.$$

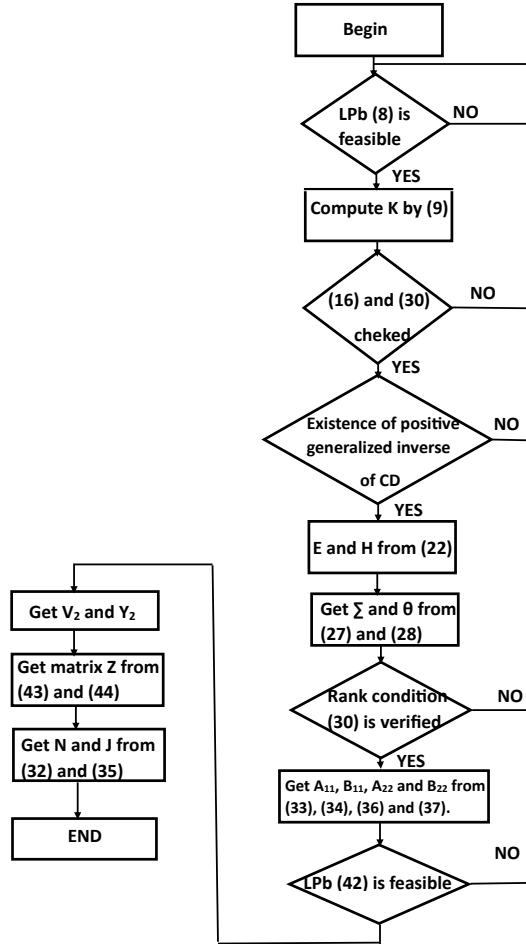


Figure 2: Algorithm block diagram for the computation of the functional observer-based control

One can verify that the closed-loop system is asymptotically stable and positive, where $K \succ 0$ is the state feedback gain.

6.2. Positive observer-based controller synthesis

After checking conditions(16) and (30), with the existence of a nonnegative generalized inverse $(CD)^-$, we obtain:

$$E = \begin{bmatrix} 0.0012 & 0.0001 \\ 0.0284 & 0.0017 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0.0119 & 0.0149 \\ 1.0128 & 0.9856 \end{bmatrix}.$$

Then,

$$\Sigma = \begin{bmatrix} 0.0208 & 0.0057 & -0.0006 \\ 1.7150 & 0.1621 & -0.0162 \\ 0 & 0 & 5.0000 \\ 0 & 2.0000 & 0.1000 \end{bmatrix}, \quad \Theta = \begin{bmatrix} -0.1862 & -0.0460 & 0.0172 \\ -15.4074 & -1.3178 & 1.0520 \end{bmatrix}.$$

The proposed LP problem (42) is feasible. One such feasible solution provides:

$$1. \quad V_2 = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad Y_2 = 10^5 \times \begin{bmatrix} 0.0007 & -0.0003 \\ -0.0532 & 0.0201 \\ 4.6323 & -1.1675 \\ -0.3534 & 0.1386 \end{bmatrix}.$$

2. Then, observer-based controller matrices N and J are given as follows:

$$N = \begin{bmatrix} -24.6659 & 0.1906 \\ 17.5750 & -9.1971 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0.0006 & 0.0313 \\ 0.1818 & 0.0368 \end{bmatrix}.$$

So, the proposed design of the positive observer-based controller for positive linear system subject to unknown inputs is obtained.

Remark 3. *It is interesting to point out that the obtained observer is positive as expected, its output $\hat{u}(t) \in \mathcal{R}_+^p$; Its positivity comes from the augmented system (13) (see Definition 5).*

Simulation results are illustrated by Figures 3–5, where we present in Figure 3 the behavior of the unknown inputs $d(t)$. In Figures 4–5 we plot together the components of the functional state $u(t) = Kx(t)$ and their estimates, i.e. the responses of the proposed observer-based controller, components of $\hat{u}(t)$.

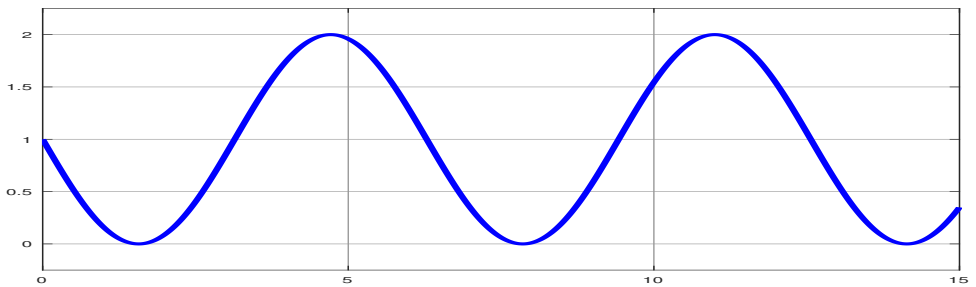


Figure 3: Evolution of unknown inputs $d(t)$ behavior

Please note that the simulations are performed with the following positive initial conditions:

$$x(0) = [2 \ 3 \ 4]^T \quad \text{and} \quad \varphi(0) = [0.2 \ 6]^T.$$

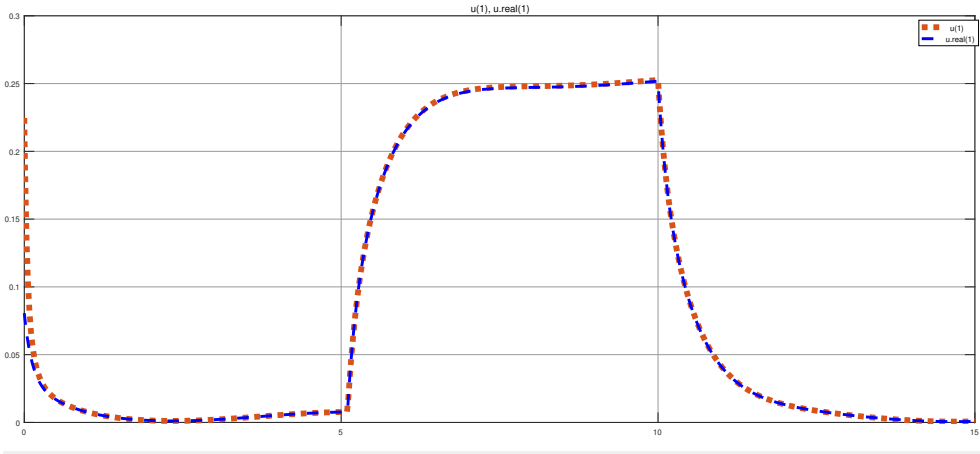


Figure 4: Functional state first component $u(1)$ with its estimated $\hat{u}(1)$

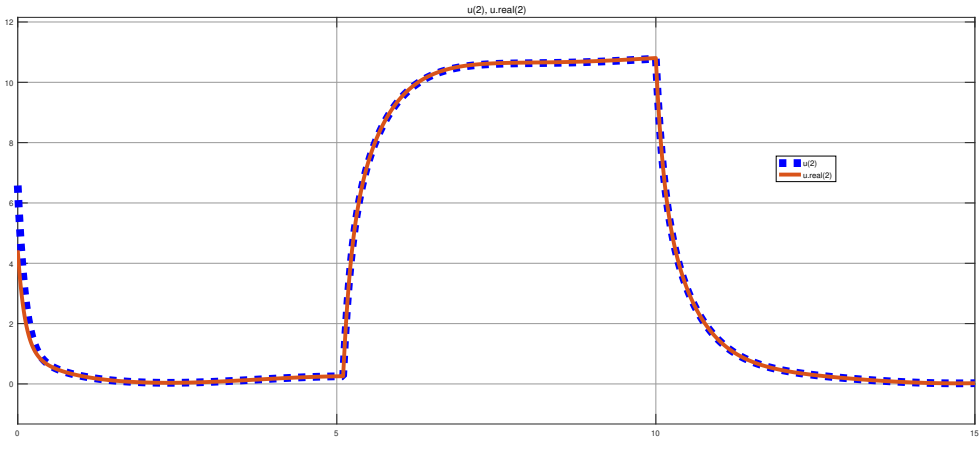


Figure 5: Functional state second component $u(2)$ with its estimated $\hat{u}(2)$

It is clear that the estimates are nonnegative and the designed observer-based controller estimated the functional state $Kx(t)$ as expected. Finally, simulation results show the behavior of the proposed positive controller based-functional observer for positive linear systems and so, the effectiveness of our approach

7. Conclusion

New results for designing positive functional observer-based controller for linear positive standard systems subject to unknown inputs are proposed in this paper. The observer is of reduced order, it is equal to the dimension of the

functional to be estimated. The positive observer-based controller is obtained into two steps. First, we search for a state feedback gain for the design of a control law such that the closed-loop is asymptotically stable and positive. Then, we search in a second step for a positive functional observer-based controller permitting to reconstruct this control law, and so to estimate only a functional of the state useful for control purposes. The proposed procedure is based on the positivity of an augmented system composed of dynamics of both considered system and proposed observer-based controller and also, on the unbiasedness of the estimation error by the resolution of Sylvester equation. Then existence conditions of such observer-based controller are formulated in terms of linear programming (LP) problem. Note that, the functional observer used has the advantages that it estimates the control law without estimating all the state of the system contrary to full order observer-based control approach. An algorithm that summarizes the different steps of the proposed controller based on positive functional observers for positive linear systems design is given.

Availability of data and materials: Not Applicable.

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