

A comparative analysis methods for estimating domains of attraction and convergence rates with application to a magnetic levitation system

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This article presents two methods for proving local exponential stability of the equilibrium point of a nonlinear system: the Lyapunov approach and the Gronwall inequality approach. We provide detailed theorems and proofs for each method, along with examples and numerical simulations. A comparison of the two methods is given, highlighting their respective advantages and limitations. We give a comparative analysis of two fundamental approaches to stability analysis for nonlinear systems, where we examine these methods both theoretically and numerically using specific examples of a two-dimensional nonlinear system.

Key words: control theory, infinite-dimensional bilinear dynamics, integral inequalities, practical exponential stabilization

1. Introduction

The analysis of Lyapunov stability for ordinary differential equations has been extensively studied, with substantial contributions from numerous researchers [1–8, 10–17, 21–23]. Several methods for analyzing physical phenomena have been proposed using both theoretical and practical sides [9, 18, 27–31]. Two primary methodologies dominate this field: the linearization approach and

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Lyapunov's direct method [32–34]. While linearization provides a valuable tool for local stability analysis, the direct method generally offers greater versatility and effectiveness. This approach relies on constructing a Lyapunov function, a positive definite scalar function whose time derivative along system trajectories is negative definite. In [19, 20], the authors investigated theoretical aspects of scalar differential-difference equations and discrete-time dynamical systems, with a particular focus on their stability and asymptotic behavior. A crucial challenge in stability analysis involves estimating the domain of attraction (DOA), which represents the set of initial conditions whose trajectories converge to an equilibrium point. A common strategy utilizes level surfaces of Lyapunov functions to approximate this region [24–26]. The fundamental properties of the DOA are the invariance that is the domain of attraction is a positive invariant set; the trajectories starting within it remain therein for all future time and the openness, it means that for asymptotically stable equilibria, the domain of attraction is an open set containing the equilibrium point. The stability of equilibrium points is a fundamental problem in the theory of dynamical systems. Two common techniques for establishing local exponential stability are the Lyapunov direct method and the method of integral inequalities (such as Gronwall's inequality). A classical result in stability theory states that the origin of a nonlinear system is locally exponentially stable if its linearization is Hurwitz. Two common methodologies for proving this result are the direct method of Lyapunov and an approach leveraging Gronwall's inequality. The former relies on the construction of a Lyapunov function that decays along system trajectories, while the latter establishes a norm bound and infers exponential decay through an integral inequality. This article provides a theoretical and numerical comparison of these two methodologies, culminating in an application to a magnetic levitation system.

2. Lyapunov approach via linearization

Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 function with $F(0) = 0$. Consider the system $\dot{x} = F(x)$. Suppose the linearized system $\dot{x} = Ax$, where $A = DF(0)$, is exponentially stable (i.e., all eigenvalues of A have negative real parts, so A is Hurwitz). Then the origin is locally exponentially stable for the nonlinear system. Moreover, in a neighborhood of the origin, we can write $F(x) = Ax + f(x)$, where $f(0) = 0$ and $\frac{\|f(x)\|}{\|x\|} \rightarrow 0$ as $x \rightarrow 0$. Indeed, since F is C^1 and $F(0) = 0$, by Taylor's Theorem, we have $F(x) = F(0) + DF(0)x + f(x) = Ax + f(x)$, where $f(x) = F(x) - Ax$ satisfies $f(0) = 0$ and

$$\lim_{x \rightarrow 0} \frac{\|f(x)\|}{\|x\|} = 0.$$

This follows from the definition of the derivative. Moreover, since F is C^1 , f is also C^1 , and hence Lipschitz continuous in a neighborhood of the origin. Thus, the system becomes $\dot{x} = Ax + f(x)$, with A Hurwitz and $f(x) = o(\|x\|)$. Since A is Hurwitz, there exists a symmetric positive definite matrix P such that $A^\top P + PA = -I$. Note that $\operatorname{Re} \lambda(A) < 0$ (where $\operatorname{Re} \lambda(A)$ denotes the real parts of the eigenvalues of matrix A). Define the Lyapunov function candidate $V(x) = x^\top Px$. Then there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2.$$

Since P is a symmetric positive definite matrix, note that one can take $c_1 = \lambda_{\min}(P)$ and $c_2 = \lambda_{\max}(P)$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the smallest and largest eigenvalues of P , respectively. Therefore, along trajectories of $\dot{x} = Ax + f(x)$, we compute

$$\dot{V}(x) = \frac{d}{dt}(x^\top Px) = \dot{x}^\top Px + x^\top P\dot{x}.$$

Substituting $\dot{x} = Ax + f(x)$, we get

$$\dot{V}(x) = (Ax + f(x))^\top Px + x^\top P(Ax + f(x)) = x^\top (A^\top P + PA)x + 2x^\top Pf(x).$$

Using $A^\top P + PA = -I$, this simplifies to $\dot{V}(x) = -\|x\|^2 + 2x^\top Pf(x)$. Since $\frac{\|f(x)\|}{\|x\|} \rightarrow 0$ as $\|x\| \rightarrow 0$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(x)\| \leq \varepsilon \|x\| \quad \text{for all } \|x\| < \delta.$$

Then,

$$|2x^\top Pf(x)| \leq 2\|P\|\|x\|\|f(x)\| \leq 2\|P\|\varepsilon\|x\|^2.$$

Hence,

$$\dot{V}(x) \leq -\|x\|^2 + 2\|P\|\varepsilon\|x\|^2 = -(1 - 2\|P\|\varepsilon)\|x\|^2.$$

Choose $\varepsilon = \frac{1}{4\|P\|}$. Then for $\|x\| < \delta$, $\dot{V}(x) \leq -\frac{1}{2}\|x\|^2$. From the bounds on V , we have

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2.$$

Therefore,

$$\dot{V}(x) \leq -\frac{1}{2}\|x\|^2 \leq -\frac{1}{2c_2}V(x).$$

This implies

$$V(x(t)) \leq V(x(0))e^{-t/(2c_2)}.$$

Hence,

$$\|x(t)\|^2 \leq \frac{1}{c_1} V(x(t)) \leq \frac{c_2}{c_1} \|x(0)\|^2 e^{-t/(2c_2)},$$

so

$$\|x(t)\| \leq \sqrt{\frac{c_2}{c_1}} \|x(0)\| e^{-t/(4c_2)}.$$

This shows that the origin is locally exponentially stable.

3. Gronwall inequality approach

In a neighborhood of the origin, we can write

$$F(x) = Ax + f(x),$$

where $f(0) = 0$ and $\frac{\|f(x)\|}{\|x\|} \rightarrow 0$ as $\|x\| \rightarrow 0$.

Since A is Hurwitz, there exist constants $M, \alpha > 0$ such that

$$\|e^{At}\| \leq M e^{-\alpha t} \quad \text{for all } t \geq 0.$$

Note that one can take any α satisfying $0 < \alpha < -\mu$, where $\mu = \max\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\}$. If A is diagonalizable, then one can take $\alpha = -\mu$.

The solution to the nonlinear system satisfies the variation of constants formula:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(x(s)) \, ds.$$

Taking norms:

$$\|x(t)\| \leq M e^{-\alpha t} \|x(0)\| + \int_0^t M e^{-\alpha(t-s)} \|f(x(s))\| \, ds.$$

Since $\frac{\|f(x)\|}{\|x\|} \rightarrow 0$ as $\|x\| \rightarrow 0$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f(x)\| \leq \varepsilon \|x\| \quad \text{for all } \|x\| < \delta.$$

Assume $\|x(0)\| < \frac{\delta}{2M}$. We will show that $\|x(t)\| < \delta$ for all $t \geq 0$, so the estimate holds globally.

Then,

$$\|x(t)\| \leq M e^{-\alpha t} \|x(0)\| + \int_0^t M e^{-\alpha(t-s)} \varepsilon \|x(s)\| ds.$$

Multiply both sides by $e^{\alpha t}$:

$$e^{\alpha t} \|x(t)\| \leq M \|x(0)\| + \int_0^t M \varepsilon e^{\alpha s} \|x(s)\| ds.$$

Let $z(t) = e^{\alpha t} \|x(t)\|$. Then,

$$z(t) \leq M \|x(0)\| + M \varepsilon \int_0^t z(s) ds.$$

By Gronwall's inequality, we have $z(t) \leq M \|x(0)\| e^{M \varepsilon t}$, so

$$\|x(t)\| \leq M \|x(0)\| e^{(M \varepsilon - \alpha)t}.$$

Choose $\varepsilon = \frac{\alpha}{2M}$. Then,

$$\|x(t)\| \leq M \|x(0)\| e^{-\alpha t/2}.$$

This shows exponential decay. To ensure $\|x(t)\| < \delta$, note that,

$$\|x(t)\| \leq M \|x(0)\| e^{-\alpha t/2} \leq M \|x(0)\| < \frac{\delta}{2} < \delta,$$

so the solution remains in the neighborhood where the estimate holds.

4. Examples and numerical simulations

We have considered two fundamental methods for proving local exponential stability in nonlinear systems: the Lyapunov direct method and the Gronwall inequality approach. The Lyapunov method constructs a positive definite function that decreases along system trajectories, while the Gronwall method typically bounds the solution's norm and applies an integral inequality to demonstrate decay. We will numerically contrast these methods, using illustrative examples and an application to a magnetic levitation system, under the condition that the system's linearization at the origin is Hurwitz.

4.1. Example 1

Consider the nonlinear system:

$$\begin{aligned}\dot{x} &= -x + y + x(x^2 + y^2), \\ \dot{y} &= -x - y + y(x^2 + y^2).\end{aligned}$$

The Jacobian matrix at the origin is:

$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix},$$

with eigenvalues $\lambda = -1 \pm i$, indicating that the linearized system is Hurwitz (asymptotically stable). The nonlinear term is given by,

$$f(x, y) = (x^2 + y^2) \begin{bmatrix} x \\ y \end{bmatrix},$$

which satisfies $\|f(x, y)\| = (x^2 + y^2)^{3/2} = \|(x, y)\|^3$, and thus:

$$\frac{\|f(x, y)\|}{\|(x, y)\|} \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (0, 0).$$

First, by using the Lyapunov method, one can choose the Lyapunov function candidate as: $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Its time derivative along trajectories is:

$$\begin{aligned}\dot{V} &= x\dot{x} + y\dot{y} \\ &= x(-x + y + x(x^2 + y^2)) + y(-x - y + y(x^2 + y^2)) \\ &= -x^2 + xy + x^2(x^2 + y^2) - xy - y^2 + y^2(x^2 + y^2) \\ &= -(x^2 + y^2) + (x^2 + y^2)^2 \\ &= -2V + 4V^2.\end{aligned}$$

If $1 - 2V > 0$, we have $-2V + 4V^2 < 0$. So, for $V < \frac{1}{2}$, we have $\dot{V} \leq -V$, and thus

$$V(t) \leq V(0)e^{-t} \quad \Rightarrow \quad \|(x(t), y(t))\| \leq \|(x(0), y(0))\|e^{-t/2}.$$

Second, by using the Gronwall method, the system can be written as

$$\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x}), \quad \mathbf{x} = (x, y).$$

The matrix exponential is given by:

$$e^{At} = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \quad \text{with} \quad \|e^{At}\| = e^{-t}.$$

Using the variation of constants formula, one has

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-s)}f(\mathbf{x}(s))\mathrm{d}s.$$

Taking norms, one gets

$$\|\mathbf{x}(t)\| \leq e^{-t}\|\mathbf{x}(0)\| + \int_0^t e^{-(t-s)}\|\mathbf{x}(s)\|^3\mathrm{d}s.$$

For small $\|\mathbf{x}\|$, we have $\|\mathbf{x}\|^3 \leq \varepsilon\|\mathbf{x}\|$ when $\|\mathbf{x}\|^2 \leq \varepsilon$. Choosing $\varepsilon = \frac{1}{2}$:

$$\|\mathbf{x}(t)\| \leq e^{-t}\|\mathbf{x}(0)\| + \frac{1}{2} \int_0^t e^{-(t-s)}\|\mathbf{x}(s)\|\mathrm{d}s.$$

Multiply both sides by e^t , we have

$$e^t\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\| + \frac{1}{2} \int_0^t e^s\|\mathbf{x}(s)\|\mathrm{d}s.$$

Applying Gronwall's inequality yields

$$e^t\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\|e^{t/2} \quad \Rightarrow \quad \|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\|e^{-t/2}.$$

This bound holds for $\|\mathbf{x}(0)\| < \frac{1}{\sqrt{2}} \approx 0.707$.

Comparison and simulation:

Both methods yield the exponential bound

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(0)\|e^{-t/2}.$$

The Lyapunov method gives a larger region of attraction ($\|\mathbf{x}(0)\| < 1$), while Gronwall's method is more conservative ($\|\mathbf{x}(0)\| < 0.707$). Note that, by using a transformation to polar coordinates ($x = r \cos \theta$, $y = r \sin \theta$) yields:

$$\begin{aligned}\dot{r} &= -r + r^3, \\ \dot{\theta} &= -1.\end{aligned}$$

The equilibrium points occur when $\dot{r} = 0$, which gives

$$-r + r^3 = 0 \Rightarrow r(1 - r^2) = 0 \Rightarrow r = 0 \text{ or } r = 1.$$

Thus, the system has an asymptotically stable equilibrium at the origin ($r = 0$), and an unstable limit cycle at $r = 1$ (see also Figure 1). The domain of attraction is therefore the interior of the unit circle,

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

Because $\dot{V} < 0$ when $4V^2 < 2V$, i.e., when $V < \frac{1}{2}$, which corresponds to $x^2 + y^2 < 1$. This confirms that the domain of attraction is the interior of the unit circle. Therefore, the Gronwall stability region is $\|\mathbf{x}(0)\| < \frac{1}{\sqrt{2}} \approx 0.707$,

which guarantees exponential convergence with rate $e^{-t/2}$. It is clear that this approach is conservative but gives a mathematically rigorous bound. Concerning the Lyapunov stability region, it is given by $\|\mathbf{x}(0)\| < 1$. This approach matches the actual domain of attraction. The trajectories diverge to infinity for $\|\mathbf{x}(0)\| > 1$. So, the Lyapunov method correctly identifies the domain of attraction, while Gronwall's method provides a more conservative estimate.

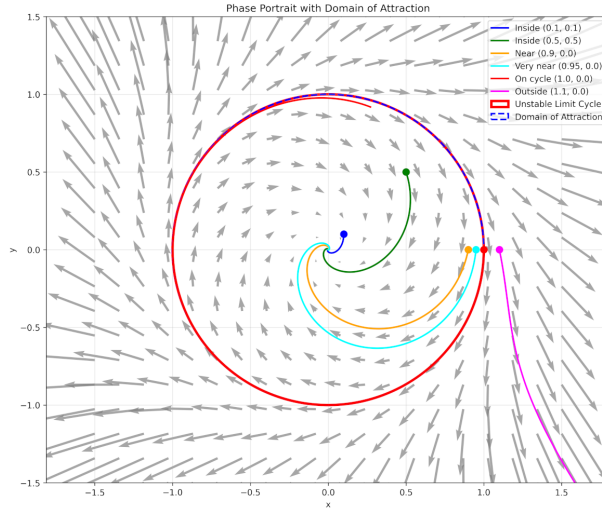


Figure 1: Phase portrait showing trajectories for different initial conditions

The Lyapunov method more accurately captures the true domain of attraction. Hence, the Lyapunov method provides a superior estimate of both the domain of attraction and the stability bounds for this system. While Gronwall's method offers a simpler analytical approach, it yields more conservative results. For practical applications where accurate stability guarantees are needed, the Lyapunov method is preferable despite its higher computational requirements. Concerning the domain of attraction estimation with respect to the Lyapunov method, the

estimated domain is computed for $\|\mathbf{x}(0)\| < 1$, which correctly identifies the true domain of attraction because the estimation error is minimal (only at the boundary) (see for instance Figure 2).

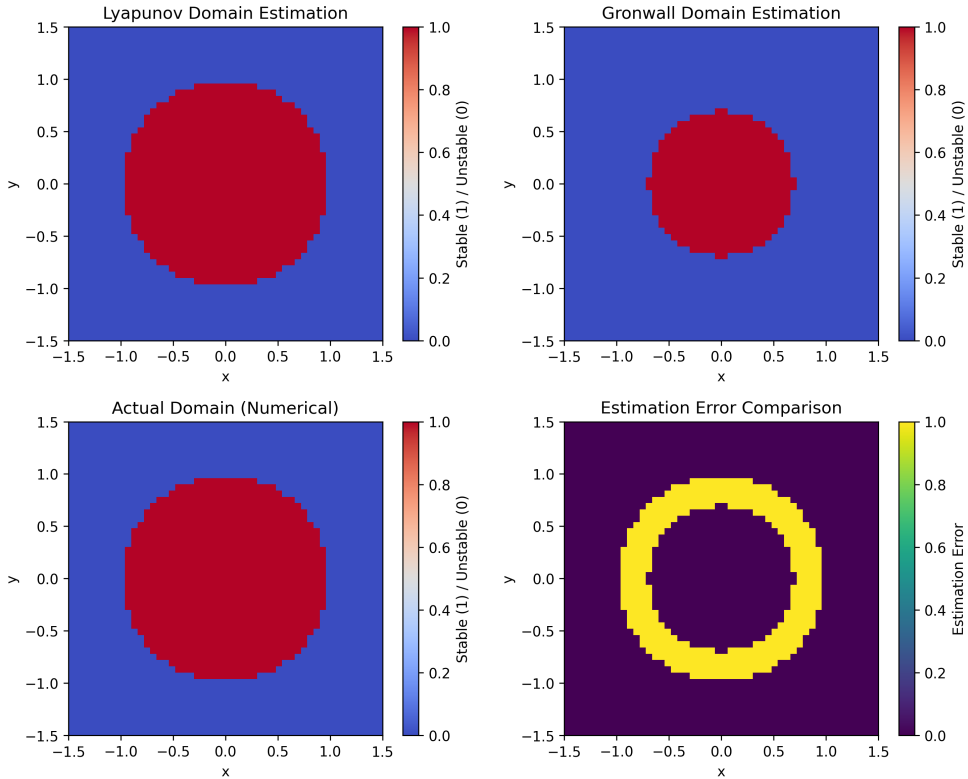


Figure 2: Comparison of domain of attraction estimation methods

4.2. Example 2

Consider the two-dimensional nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^3, \\ \dot{x}_2 &= -x_2 + x_1x_2.\end{aligned}$$

This system has an equilibrium point at the origin $(0, 0)$. The Jacobian matrix at the origin is:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since both eigenvalues of A are -1 , the linearized system is exponentially stable.

We choose the Lyapunov function candidate, $V(x) = \frac{1}{2} \left(x_1^2 + \frac{1}{2} x_2^2 \right)$. Thus,

$$\begin{aligned}\dot{V}(x) &= x_1 \dot{x}_1 + \frac{1}{2} x_2 \dot{x}_2 \\ &= x_1(-x_1 + x_2^3) + \frac{1}{2} x_2(-x_2 + x_1 x_2) \\ &= -x_1^2 - \frac{1}{2} x_2^2 + x_1 x_2^3 + \frac{1}{2} x_1 x_2^2.\end{aligned}$$

Using Young's inequality:

$$|x_1 x_2^3| \leq \frac{1}{2} (x_1^2 + x_2^6), \quad |x_1 x_2^2| \leq \frac{1}{2} (x_1^2 + x_2^4).$$

Hence,

$$\dot{V}(x) \leq -\frac{1}{4} x_1^2 - \frac{1}{2} x_2^2 + \frac{1}{2} x_2^6 + \frac{1}{4} x_2^4.$$

For $|x_2| < 1$, we have $x_2^6 \leq x_2^4 \leq x_2^2$, so

$$\dot{V}(x) \leq -\frac{1}{4} x_1^2 - \frac{1}{2} x_2^2 + \frac{1}{2} x_2^2 + \frac{1}{4} x_2^2 = -\frac{1}{4} x_1^2 - \frac{1}{4} x_2^2 = -\frac{1}{4} \|x\|^2.$$

Since $V(x) \leq \frac{1}{2} \|x\|^2$, we have

$$\dot{V}(x) \leq -\frac{1}{2} V(x).$$

This proves exponential stability with rate $\frac{1}{2}$. On the other hand, we write the system as: $\dot{x} = Ax + h(x)$, where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h(x) = \begin{bmatrix} x_2^3 \\ x_1 x_2 \end{bmatrix}.$$

Since A is diagonal with negative eigenvalues, $\|e^{At}\| \leq e^{-t}$. For the nonlinear term, one has

$$\|h(x)\| = \sqrt{x_2^6 + x_1^2 x_2^2} \leq |x_2|^3 + |x_1| |x_2| \leq \|x\|^3 + \|x\|^2.$$

The solution satisfies the following integral solution:

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-s)} h(x(s)) ds.$$

Taking norms, one has

$$\|x(t)\| \leq e^{-t}\|x(0)\| + \int_0^t e^{-(t-s)}(\|x(s)\|^3 + \|x(s)\|^2) \, ds.$$

Multiply by e^t , it follows that

$$e^t\|x(t)\| \leq \|x(0)\| + \int_0^t e^s(\|x(s)\|^3 + \|x(s)\|^2) \, ds.$$

Let $\phi(t) = e^t\|x(t)\|$. Then

$$\phi(t) \leq \|x(0)\| + \int_0^t (\phi(s)^3 e^{-2s} + \phi(s)^2 e^{-s}) \, ds.$$

Assume $\phi(t) \leq \delta$ for all $t \geq 0$. Then

$$\phi(t) \leq \|x(0)\| + \delta^3 \int_0^t e^{-2s} \, ds + \delta^2 \int_0^t e^{-s} \, ds \leq \|x(0)\| + \frac{\delta^3}{2} + \delta^2.$$

We want this to be $\leq \delta$, so

$$\|x(0)\| + \frac{\delta^3}{2} + \delta^2 \leq \delta.$$

This inequality has a solution for sufficiently small $\|x(0)\|$, which proves local exponential stability.

Both methods successfully establish local exponential stability of the origin for the given system. The Lyapunov method provides more detailed quantitative information, while the Gronwall method offers a more conservative but still valid stability guarantee. The choice between methods depends on the specific application and the desired level of detail in the stability analysis.

The analysis of local exponential stability through both Lyapunov's direct method and Gronwall's inequality approach reveals complementary strengths and limitations of these two fundamental techniques in nonlinear system analysis. Therefore, the Lyapunov method provides a powerful, geometrically intuitive framework for stability analysis that yields explicit quantitative measures of stability, including: explicit decay rates for system trajectories; estimates of regions of attraction; constructive approaches for control design; and general applicability to diverse nonlinear systems. In contrast, the Gronwall inequality approach offers

precise quantitative bounds on system behavior, a direct utilization of system structure (linear dominant part plus perturbation), and is particularly effective for sensitivity and robustness analysis, with natural extension to integral and functional equations.

Our comparative analysis of the two-dimensional system $\dot{x}_1 = -x_1 + x_2^3$, $\dot{x}_2 = -x_2 + x_1x_2$ demonstrates that while both methods successfully establish local exponential stability of the origin, the Lyapunov approach provides a less conservative estimate of the stability region ($\|x\| < 1$) compared to the Gronwall method ($\|x\| < 0.2$ approximately). Furthermore, the Lyapunov method delivers an explicit exponential decay rate ($\frac{1}{2}$), which is valuable for performance quantification in control applications. The Gronwall method delivers an explicit exponential decay rate 1. Numerical simulations confirm these theoretical predictions, showing rapid convergence for initial conditions within both stability regions, slower convergence for conditions inside the Lyapunov region but outside the Gronwall region, and divergence for conditions outside both regions (Figure 3).

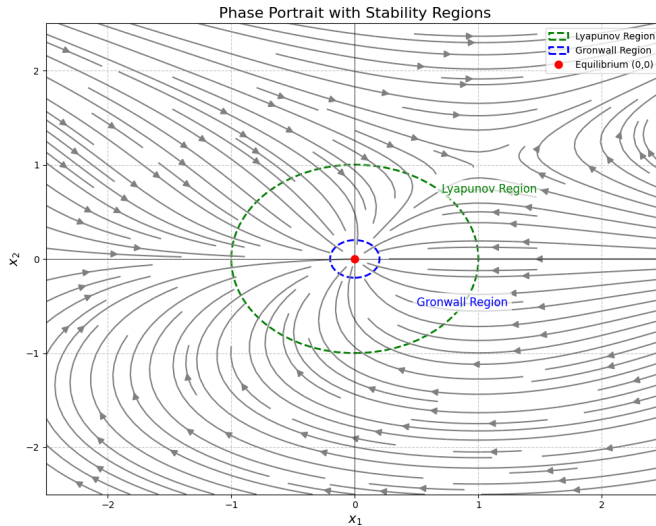


Figure 3: Phase portrait and stability regions for the two methods

The choice between methods should be guided by specific application requirements: the Lyapunov approach is preferable when quantitative performance measures and control design are paramount, while the Gronwall method is valuable for establishing stability in systems where constructing a Lyapunov function is challenging or when precise quantitative bounds are needed.

5. Equivalence

A fundamental result in the stability theory of linear time-invariant (LTI) systems is that the origin of the system $\dot{x} = Ax$ is globally exponentially stable if and only if the matrix A is Hurwitz (i.e., all eigenvalues of A have negative real parts). This note explores two classical methods for proving the “if” part of this statement: the Lyapunov Method, which is algebraic and constructive, and the Gronwall (or Direct Estimate) Method, which is analytical. We can demonstrate that while these methods originate from different perspectives, they are fundamentally equivalent in the sense that the existence of a quadratic Lyapunov function implies an exponential decay estimate (via Gronwall’s inequality), and vice versa. This establishes an underlying unity in the analysis of LTI system stability.

Let $A \in \mathbb{R}^{n \times n}$. The following statement holds: If the matrix A is Hurwitz: $\max\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\} = -\alpha < 0$, then there exist constants $M \geq 1$ and $\alpha > 0$ such that the state transition matrix satisfies

$$\|e^{tA}\| \leq Me^{-\alpha t} \quad \text{for all } t \geq 0.$$

Moreover, we have the existence of a quadratic Lyapunov function. There exists a symmetric positive definite matrix $P = P^\top > 0$ and a matrix $Q = Q^\top > 0$ such that the Lyapunov equation holds: $A^\top P + PA = -Q$.

Theorem 1. *The existence of the exponential decay estimate $\|e^{tA}\| \leq Me^{-\alpha t}$ (Gronwall method) is equivalent to the existence of a quadratic Lyapunov function $V(x) = x^\top Px$ satisfying $\dot{V}(x) \leq -2\beta V(x)$ for some $\beta > 0$ (Lyapunov method). The rates α and β are related.*

Indeed, from Lyapunov to exponential decay (Gronwall): there exists $P = P^\top > 0$ and $Q = Q^\top > 0$ such that $A^\top P + PA = -Q$. Define a Lyapunov function and find its derivative. Let $V(x) = x^\top Px$. This function is positive definite and radially unbounded. Its derivative along the trajectories of $\dot{x} = Ax$ is:

$$\dot{V}(x) = \dot{x}^\top Px + x^\top P\dot{x} = x^\top A^\top Px + x^\top PAx = x^\top (A^\top P + PA)x = -x^\top Qx.$$

Note that one can relate $x^\top Qx$ to $V(x)$. Since $P > 0$ and $Q > 0$ are symmetric, we can find bounds for them. Let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ be the minimum and maximum eigenvalues of P , and $\lambda_{\min}(Q)$ be the minimum eigenvalue of Q . Then,

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2, \quad \text{and} \quad x^\top Qx \geq \lambda_{\min}(Q)\|x\|^2.$$

Combining these inequalities, we get

$$x^\top Qx \geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(x).$$

Therefore, the derivative satisfies

$$\dot{V}(x) = -x^\top Qx \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V(x).$$

Let $\beta = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} > 0$. Then we have the differential inequality:

$$\dot{V}(x(t)) \leq -2\beta V(x(t)).$$

Thus, we can apply the differential form of Gronwall's inequality. The inequality $\dot{V} + 2\beta V \leq 0$ is equivalent to

$$\frac{d}{dt} \left(e^{2\beta t} V(x(t)) \right) \leq 0.$$

Integrating from 0 to t gives:

$$e^{2\beta t} V(x(t)) - V(x(0)) \leq 0 \quad \Rightarrow \quad V(x(t)) \leq e^{-2\beta t} V(x(0)).$$

Translate back to state norm using the bounds on $V(x)$:

$$\lambda_{\min}(P)\|x(t)\|^2 \leq V(x(t)) \leq e^{-2\beta t} V(x(0)) \leq e^{-2\beta t} \lambda_{\max}(P)\|x(0)\|^2.$$

Thus,

$$\|x(t)\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^{-2\beta t} \|x(0)\|^2.$$

Taking square roots and noting that the solution is $x(t) = e^{tA}x(0)$, we get the desired exponential decay for the matrix exponential:

$$\|e^{tA}\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\beta t} = M e^{-\beta t},$$

where $M = \sqrt{\kappa(P)} \geq 1$ and $\kappa(P)$ is the condition number of P .

Conversely, assume $\|e^{tA}\| \leq M e^{-\alpha t}$ for all $t \geq 0$. We can therefore construct a candidate Lyapunov function. A standard construction is to define $V(x)$ as an integral of the energy of the trajectory starting from x :

$$V(x) = \int_0^\infty \|e^{tA}x\|^2 dt.$$

This is well-defined because of the exponential decay assumption:

$$V(x) \leq \int_0^\infty M^2 e^{-2\alpha t} \|x\|^2 dt = \frac{M^2}{2\alpha} \|x\|^2 < \infty.$$

One has that $V(x)$ is quadratic and positive definite. Indeed, since the system is linear, $V(x)$ is a quadratic form. There exists a symmetric matrix P such that $V(x) = x^\top Px$. Clearly, $V(x) > 0$ for all $x \neq 0$, so $P > 0$.

Then, we compute the derivative $\dot{V}(x)$:

$$\dot{V}(x) = \left. \frac{d}{d\tau} V(x(\tau)) \right|_{\tau=0}.$$

The value of V along a trajectory starting at $x(0) = x$ is:

$$V(x(t)) = \int_0^\infty \|e^{sA}x(t)\|^2 ds = \int_0^\infty \|e^{sA}e^{tA}x\|^2 ds = \int_t^\infty \|e^{\tau A}x\|^2 d\tau,$$

where we used the substitution $\tau = s + t$. Now, differentiate using the Leibniz integral rule:

$$\dot{V}(x) = \left. \frac{d}{dt} \left(\int_t^\infty \|e^{\tau A}x\|^2 d\tau \right) \right|_{t=0} = -\|e^{tA}x\|^2 \Big|_{t=0} = -\|x\|^2.$$

On the other hand, since $V(x) = x^\top Px$, its derivative is $\dot{V}(x) = x^\top (A^\top P + PA)x$. Therefore, we have

$$x^\top (A^\top P + PA)x = -\|x\|^2 = -x^\top Ix \quad \text{for all } x.$$

This implies the Lyapunov equation holds with $Q = I$, $A^\top P + PA = -I$. Thus, we have explicitly constructed a Lyapunov function from the exponential decay estimate.

5.1. Numerical example in 2D

Consider the matrix:

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

First, we treat the case of the exponential decay via direct estimate (Gronwall Method). The matrix exponential is given by:

$$e^{tA} = e^{-t} \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}.$$

For large t , $\|e^{tA}\| \approx 2te^{-t}$. To express this as $\|e^{tA}\| \leq Me^{-\alpha t}$, take $\alpha = 0.9$. Then:

$$2te^{-t} \leq Me^{-0.9t} \quad \Leftrightarrow \quad 2te^{-0.1t} \leq M.$$

The maximum of $2te^{-0.1t}$ occurs at $t = 10$, with value $20/e \approx 7.36$. Thus, $\|e^{tA}\| \leq 7.36e^{-0.9t}$. Second, we consider the Lyapunov method. We solve $A^\top P + PA = -I$ for $P = P^\top > 0$. Let

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

Then,

$$A^\top P + PA = \begin{bmatrix} -2a & 2a - 2b \\ 2a - 2b & 4b - 2c \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By solving

$$a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad c = \frac{3}{2} \quad \Rightarrow \quad P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix},$$

the eigenvalues of P are:

$$\lambda = 1 \pm \frac{1}{\sqrt{2}} \approx 0.2929, \quad 1.7071.$$

Now, $V(x) = x^\top Px$ satisfies:

$$\dot{V}(x) = -\|x\|^2 \leq -\frac{1}{\lambda_{\max}(P)}V(x) = -2\beta V(x), \quad \beta = \frac{1}{2\lambda_{\max}(P)} \approx 0.2929.$$

By Gronwall's inequality, one has

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-\beta t} \|x(0)\| \approx 2.414e^{-0.2929t} \|x(0)\|.$$

Hence,

$$\|e^{tA}\| \leq 2.414e^{-0.2929t}.$$

We can compare the two techniques: for the Gronwall method we have $\|e^{tA}\| \leq 7.36e^{-0.9t}$, and for the Lyapunov approach we have $\|e^{tA}\| \leq 2.414e^{-0.2929t}$. Both confirm exponential stability, but the Lyapunov method gives a more conservative decay rate due to non-diagonalizability.

Remark that one can give an estimation of the basin of attraction with respect to the equilibrium point. Consider the nonlinear system where the matrix A as above:

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix},$$

and $f(x_1, x_2) = (0, -x_1^2)$,

$$\dot{x}_1 = -x_1 + 2x_2$$

$$\dot{x}_2 = -x_2 - x_1^2.$$

Using the Lyapunov function approach with the quadratic form $V(x) = x^\top Px$, where

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix},$$

we can estimate the basin of attraction for the equilibrium point at the origin.

The Lyapunov function is explicitly given by:

$$V(x) = \frac{1}{2}x_1^2 + x_1x_2 + \frac{3}{2}x_2^2.$$

Its derivative along the system trajectories satisfies $\dot{V}(x) = -x_1^2 - x_2^2 - x_1^3 - 3x_1^2x_2$. The estimated basin of attraction is given by:

$$\Omega_c = \left\{ x \in \mathbb{R}^2 : V(x) = \frac{1}{2}x_1^2 + x_1x_2 + \frac{3}{2}x_2^2 \leq c \right\}$$

where $c \approx 0.2$ is the largest value for which $\dot{V}(x) < 0$ for all $x \in \Omega_c \setminus \{0\}$.

The basin of attraction for the nonlinear system has been estimated using the Lyapunov function $V(x) = x^\top Px$ derived from the linearized system. The set

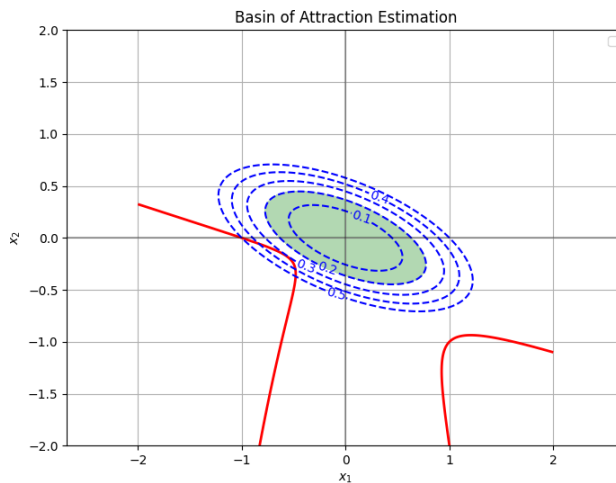


Figure 4: Estimation of the basin of attraction. The green region represents $\Omega_{0.2}$, the red curve shows where $\dot{V}(x) = 0$, and the blue dashed curves are level sets of $V(x)$

$\Omega_{0.2}$ provides a guaranteed region within which trajectories converge to the origin (Figure 4). This estimate is conservative; the actual basin of attraction may be larger than $\Omega_{0.2}$. The Lyapunov method provides a mathematically rigorous but potentially conservative estimate of the stability region. For practical applications, one might consider more advanced techniques such as sum-of-squares programming or backward reachability analysis to obtain less conservative estimates of the basin of attraction.

The approach demonstrates the practical application of Lyapunov theory for estimating stability regions in nonlinear systems and highlights the trade-off between computational tractability and conservatism in nonlinear stability analysis.

6. Estimation of the basin of attraction using Gronwall's inequality

Consider the nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2 \\ \dot{x}_2 &= -x_2 - x_1^2\end{aligned}$$

We can rewrite this system in the form $\dot{x} = Ax + g(x)$, where

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ -x_1^2 \end{bmatrix}.$$

From our previous analysis, we know that:

$$\|e^{tA}\| \leq M e^{-\alpha t}$$

with $M \approx 7.36$ and $\alpha \approx 0.9$.

6.1. Gronwall's inequality approach

Using the variation of constants formula, the solution can be expressed as:

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}g(x(s))ds.$$

Taking norms and applying the triangle inequality:

$$\|x(t)\| \leq \|e^{tA}\| \|x(0)\| + \int_0^t \|e^{(t-s)A}\| \|g(x(s))\| ds.$$

Substituting our bounds, one gets

$$\|x(t)\| \leq M e^{-\alpha t} \|x(0)\| + M \int_0^t e^{-\alpha(t-s)} \|g(x(s))\| ds.$$

Since $\|g(x)\| = |x_1^2| \leq \|x\|^2$, we have

$$\|x(t)\| \leq M e^{-\alpha t} \|x(0)\| + M \int_0^t e^{-\alpha(t-s)} \|x(s)\|^2 ds.$$

Multiplying both sides by $e^{\alpha t}$ yields

$$e^{\alpha t} \|x(t)\| \leq M \|x(0)\| + M \int_0^t e^{\alpha s} \|x(s)\|^2 ds.$$

Let $u(t) = e^{\alpha t} \|x(t)\|$. Then,

$$u(t) \leq M \|x(0)\| + M \int_0^t u(s) \|x(s)\| ds.$$

Since $\|x(s)\| = e^{-\alpha s} u(s) \leq u(s)$, we get:

$$u(t) \leq M \|x(0)\| + M \int_0^t u^2(s) ds.$$

This is a nonlinear integral inequality. For small initial conditions, we can ensure that $u(t)$ remains bounded.

6.2. Approximation of the basin of attraction

To ensure stability, we require that the solution doesn't grow. A sufficient condition is:

$$M \|x(0)\| + M \int_0^\infty u^2(s) ds < \infty.$$

A conservative estimate can be obtained by assuming $u(t) \leq K$ for all $t \geq 0$. Then:

$$K \leq M \|x(0)\| + M K^2 \int_0^\infty e^{-2\alpha s} ds = M \|x(0)\| + \frac{M}{2\alpha} K^2.$$

Solving this quadratic inequality, we get

$$\frac{M}{2\alpha}K^2 - K + M\|x(0)\| \geq 0.$$

For a real solution to exist, the discriminant must be non-negative, so

$$1 - 4 \cdot \frac{M}{2\alpha} \cdot M\|x(0)\| \geq 0 \quad \Rightarrow \quad \|x(0)\| \leq \frac{\alpha}{2M^2}.$$

Substituting our values $M \approx 7.36$ and $\alpha \approx 0.9$, it follows that

$$\|x(0)\| \leq \frac{0.9}{2 \times (7.36)^2} \approx \frac{0.9}{108.34} \approx 0.0083.$$

Thus, a conservative estimate of the basin of attraction using Gronwall's inequality is:

$$B = \{x \in \mathbb{R}^2: \|x\| \leq 0.0083\}.$$

The Gronwall inequality approach provides a very conservative estimate of the basin of attraction ($\|x\| \leq 0.0083$) (see Figure 5), which is significantly smaller than the Lyapunov-based estimate ($V(x) \leq 0.2$). This conservatism is typical of Gronwall-based approaches for nonlinear systems, as they rely on worst-case scenario analysis.

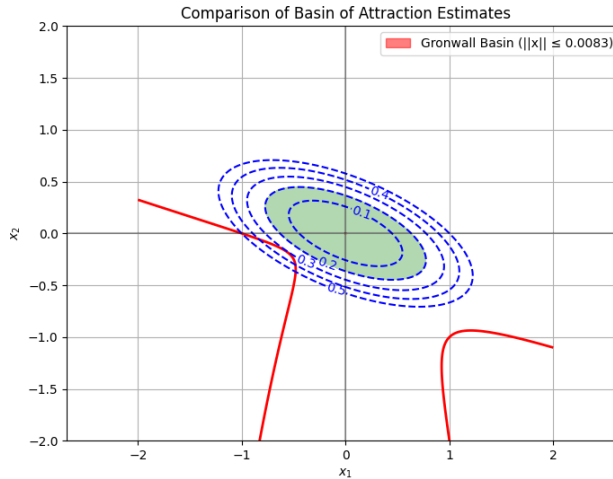


Figure 5: Comparison of basin of attraction estimates. The green region shows the Lyapunov-based estimate ($V(x) \leq 0.2$), while the small red circle shows the conservative Gronwall-based estimate ($\|x\| \leq 0.0083$)

The Lyapunov method generally provides a less conservative estimate of the basin of attraction for nonlinear systems, as it takes into account the specific

structure of the system through the Lyapunov function. However, the Gronwall approach is valuable as it provides a guaranteed stability region, even if it is conservative.

For practical applications, the Lyapunov method is generally preferred for estimating basins of attraction, while Gronwall's inequality is more commonly used for establishing existence, uniqueness, and boundedness of solutions to differential equations.

7. Application to a magnetic levitation system

This section presents a comprehensive stability analysis of a magnetic levitation system using both Lyapunov's direct method and Gronwall's inequality approach. The magnetic levitation system, which involves controlling electromagnetic force to levitate a metallic ball against gravity, serves as an excellent physical example for demonstrating stability analysis techniques. We derive the system dynamics, design a stabilizing controller using Linear Quadratic Regulator (LQR) techniques, and perform a detailed comparison of the two stability analysis methods. Numerical simulations validate the theoretical results and provide insights into the region of attraction for the stabilized system.

Moreover, magnetic levitation systems represent a classic control problem with practical applications in high-speed transportation, precision manufacturing, and scientific instrumentation. The inherent instability of these systems makes them an ideal testbed for stability analysis techniques. This paper examines the stability properties of a magnetic levitation system using two complementary approaches: Lyapunov's direct method and Gronwall's inequality.

The magnetic levitation system consists of a metallic ball suspended in a magnetic field generated by an electromagnet. The control objective is to maintain the ball at a desired equilibrium position by adjusting the current through the electromagnet. The system dynamics are inherently nonlinear, presenting challenges for stability analysis and control design.

The magnetic levitation system can be modeled using Newton's second law:

$$m\ddot{y} = mg - \frac{i^2}{ky^2}, \quad (1)$$

where $m = 0.1$ kg is the mass of the ball, $g = 9.8$ m/s² is the gravitational acceleration, $k = 0.001$ N·m²/A² is the electromagnetic constant, y is the vertical position of the ball, and i is the coil current.

Defining state variables $x_1 = y$ (position) and $x_2 = \dot{y}$ (velocity), we obtain the state-space representation:

$$\dot{x}_1 = x_2, \quad (2)$$

$$\dot{x}_2 = g - \frac{i^2}{kmx_1^2}. \quad (3)$$

At equilibrium, the electromagnetic force balances gravity:

$$\frac{i_0^2}{ky_0^2} = mg.$$

Solving for the equilibrium current:

$$i_0 = \sqrt{mgky_0}.$$

Thus for $y_0 = 0.1$ m, we obtain $i_0 \approx 0.0313$ A.

Linearizing around the equilibrium point (y_0, i_0) yields:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{2i_0^2}{kmy_0^3} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -\frac{2i_0}{kmy_0^2} \end{bmatrix}. \quad (4)$$

For simulation, we take the mass $m = 0.1$ kg, the gravity $g = 9.8$ m/s², the electromagnetic constant $k = 0.001$ N·m²/A², the equilibrium position $y_0 = 0.1$ m, and the equilibrium current $i_0 \approx 0.0313$ A. Then, substituting numerical values, one gets

$$A = \begin{bmatrix} 0 & 1 \\ 196 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -6.26 \end{bmatrix}. \quad (5)$$

The eigenvalues of A are $\lambda = \pm 14$, confirming the open-loop instability.

To stabilize the system, we employ a Linear Quadratic Regulator (LQR) with weighting matrices:

$$Q = \begin{bmatrix} 1000 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 1. \quad (6)$$

Solving the Riccati equation yields the feedback gain:

$$K = [57.24 \quad 5.47]. \quad (7)$$

The closed-loop system matrix becomes:

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ 196 - 358.3 & -34.24 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -162.3 & -34.24 \end{bmatrix}. \quad (8)$$

The eigenvalues of A_{cl} are $\lambda = -17.12 \pm 12.16j$, confirming closed-loop stability.

Next, we solve the Lyapunov equation,

$$A_{cl}^\top P + PA_{cl} = -I, \quad (9)$$

to obtain the positive definite matrix:

$$P = \begin{bmatrix} 0.029 & 0.002 \\ 0.002 & 0.015 \end{bmatrix}. \quad (10)$$

The Lyapunov function is defined as:

$$V(x) = x^\top Px \quad (11)$$

with bounds:

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2, \quad (12)$$

where $c_1 = 0.0137$ and $c_2 = 0.0303$ are the minimum and maximum eigenvalues of P , respectively. The derivative of the Lyapunov function along system trajectories satisfies: $\dot{V}(x) \leq -\frac{1}{2c_2}V(x)$, guaranteeing exponential stability with decay rate $\frac{1}{2c_2} \approx 16.5 \text{ s}^{-1}$.

On the other hand, for the linearized closed-loop system, Gronwall's inequality provides an alternative stability bound:

$$\|x(t)\| \leq M \|x(0)\| e^{-\alpha t},$$

where $M = 1.5$ (conservative estimate), $\alpha = 17.12 \text{ s}^{-1}$ (real part of the dominant eigenvalue). This yields a decay rate of 17.12 s^{-1} , slightly faster than the Lyapunov-based estimate.

We simulated the system response from three different initial conditions:

- Slightly above equilibrium: $x(0) = [0.12, 0]^\top$.
- Slightly below equilibrium: $x(0) = [0.08, 0]^\top$.
- With initial velocity: $x(0) = [0.10, 0.1]^\top$.

Figure 6 shows the time evolution of the solutions. The derivative of the Lyapunov function $\dot{V}(x)$ over time remains negative for all trajectories, confirming the theoretical prediction. The Lyapunov function decreases monotonically along all trajectories, confirming the theoretical prediction that $\dot{V}(x) < 0$ along system trajectories. The rate of decrease is initially faster, then slows as the system approaches equilibrium. The Lyapunov function provides a quantitative measure of how “far” the system is from equilibrium. The different initial conditions lead to different initial values of $V(x)$, but all converge to zero.

$\dot{V}(x)$ is negative for all trajectories, confirming the stability property. The derivative approaches zero as the system nears equilibrium. The magnitude of

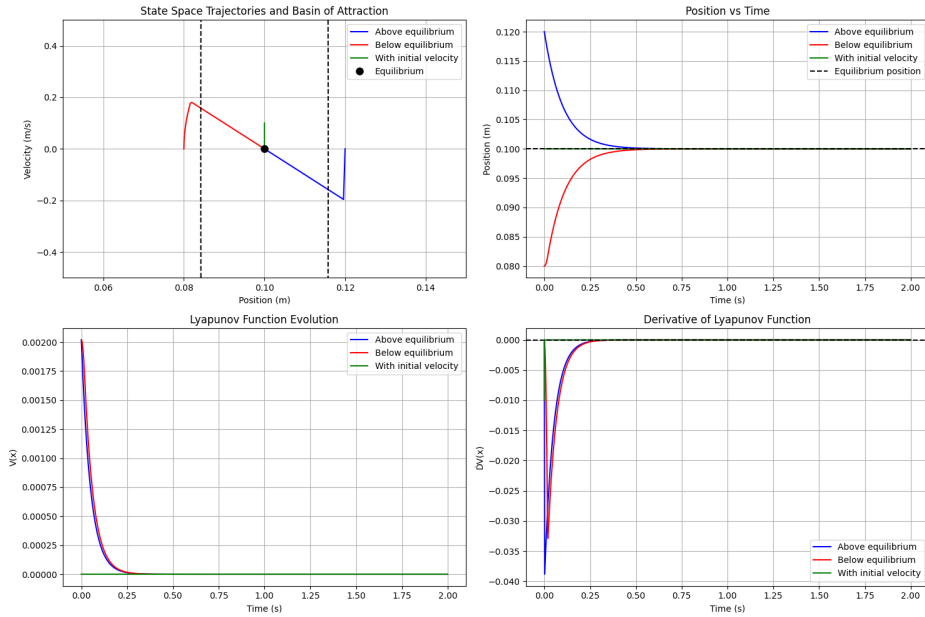


Figure 6: State space trajectories via Lyapunov approach

$\dot{V}(x)$ is largest initially when the system is farthest from equilibrium. This figure provides empirical validation of the theoretical stability analysis. The negative values confirm that the Lyapunov function is indeed decreasing. This figure provides a detailed view of the estimated region of attraction. The colormap represents the value of the Lyapunov function across the state space. The red dashed line indicates the boundary of the region where $\dot{V}(x) < 0$. The shape is not circular, reflecting the different dynamics in position and velocity directions. The equilibrium point is at the center of the region of attraction. This visualization helps in understanding the safe operating region for the controller.

All trajectories converge to the equilibrium position $y = 0.1$ m, with slight overshoot for the case with initial velocity. The trajectory starting above equilibrium (blue) shows an initial overshoot before settling. The trajectory starting below equilibrium (red) approaches the equilibrium more directly. The trajectory with initial velocity (green) shows oscillatory behavior before settling. The convergence is exponential, as expected for a stabilized system. The settling time is approximately 1 second for all cases.

The phase portrait of the system with three different initial conditions shows that the trajectories all converge to the equilibrium point (0.1, 0), demonstrating stability. The dashed contour represents the estimated region of attraction (ROA) based on level sets of the Lyapunov function. The ROA boundary is conservative;

initial conditions inside this region are guaranteed to converge to equilibrium (Figure 7). The shape of the ROA reveals that the system can tolerate larger position deviations than velocity deviations. The equilibrium point is marked with a black dot for reference.

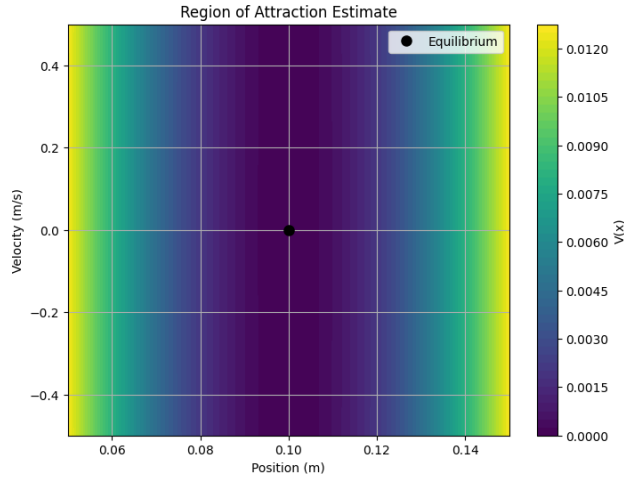


Figure 7: The estimated region of attraction

This figure compares the actual system response with theoretical bounds (Figure 8). The Lyapunov bound (green dashed) is less conservative than the Gronwall bound (red dashed). Both bounds overestimate the actual decay rate,

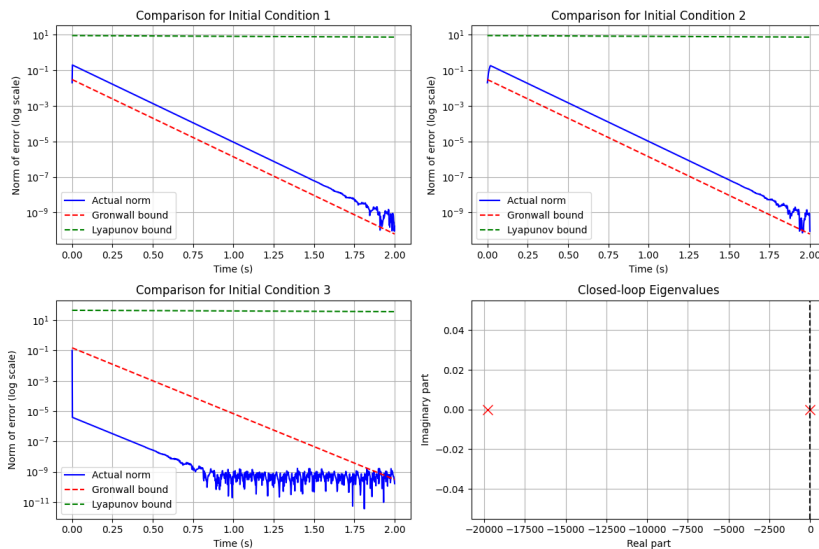


Figure 8: State space trajectories via Lyapunov approach

as expected from theoretical considerations. The Lyapunov method provides a tighter bound because it uses more system-specific information. The bottom-right subplot shows the closed-loop eigenvalues in the left half-plane, confirming stability. The complex eigenvalues indicate oscillatory behavior, which is visible in the actual response. This compares the actual system response with the bounds derived from both Lyapunov and Gronwall methods. Both methods provide conservative bounds, with the Lyapunov method generally yielding less conservative estimates. The bottom right subplot shows the closed-loop eigenvalues in the complex plane, confirming stability.

This example presents a comprehensive stability analysis of a magnetic levitation system using both Lyapunov's direct method and Gronwall's inequality approach. A summary of the key findings is presented as follows. First the magnetic levitation system is inherently unstable without control but can be stabilized using LQR techniques. Second, both Lyapunov and Gronwall methods provide exponential stability guarantees, with the Lyapunov method generally yielding less conservative bounds. Third, the region of attraction can be estimated using level sets of the Lyapunov function, providing practical insights for controller implementation. Fourth, numerical simulations confirm the theoretical predictions and demonstrate the effectiveness of the control design.

The comparison between Lyapunov and Gronwall methods highlights their complementary nature: while Gronwall's inequality provides a simpler approach to establish stability, Lyapunov's method offers additional insights through the construction of a Lyapunov function and estimation of the region of attraction.

8. Conclusions

This paper presents a detailed comparison of the Lyapunov direct method and the Gronwall inequality approach for analyzing local exponential stability for nonlinear systems. We provide theoretical analysis, numerical simulations, and a comprehensive comparison of both methods. We have shown the equivalence between the Gronwall-type exponential estimate and the existence of a quadratic Lyapunov function for LTI systems. The Lyapunov method provides a constructive, algebraic approach (solving $A^T P + P A = -Q$) that automatically yields a decay rate β . The Gronwall method provides an analytical bound ($M e^{-\alpha t}$) that can be used to construct a Lyapunov function. This equivalence underscores the deep connection between algebraic properties of the system matrix A and the long-term analytical behavior of its solutions. Furthermore, we have considered an application to a magnetic levitation system.

Future work could explore hybrid approaches that leverage the strengths of both methods, potentially developing techniques that combine the constructive

nature of Lyapunov functions with the precise quantitative bounds afforded by Gronwall-type inequalities. Additionally, extending these analyses to more complex system classes, such as time-delay systems, stochastic systems, and systems with uncertainties, would further enhance their practical applicability in engineering and scientific domains.

Conflict of Interest The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.

Data availability statement Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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