

Observer-based practical stabilization of fractional-order systems via linear matrix inequalities

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This paper investigates the problem of observer-based stabilization for a class of nonlinear fractional-order systems affected by bounded disturbances. The considered nonlinearities satisfy the quasi-one-sided Lipschitz condition, which generalizes the classical one-sided Lipschitz property and allows treating a broader class of nonlinear dynamics that are not necessarily Lipschitz continuous. First, sufficient conditions ensuring the global uniform practical Mittag-Leffler stability of both the observer error dynamics and the state-feedback closed-loop system are derived in terms of Linear Matrix Inequalities (LMIs). Then, an observer-based output feedback controller is designed by combining the state-feedback law and the fractional-order observer. The proposed separation principle guarantees the practical Mittag-Leffler stability of the overall closed-loop system. An illustrative numerical example is provided to demonstrate the effectiveness and reduced conservatism of the proposed design.

Key words: fractional-order systems, quasi-one-sided Lipschitz condition, observer-based control, Mittag-Leffler stability, Linear Matrix Inequalities (LMI), disturbance attenuation

1. Introduction

Fractional-order operators have emerged as a powerful modelling tool for physical systems with memory-dependent dynamics, interpolating between successive integer derivatives and encoding the complete history of state evolution through a power-law convolution kernel. Their applications span electrochemical cells [15], dielectric relaxation [20], viscoelastic damping [12, 19], polymer mechanics [3], anomalous diffusion [6], and wave propagation [11], motivating

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extensive research into fractional-order control synthesis and state reconstruction [10, 21, 24].

Within nonlinear control theory, Lipschitz-type conditions provide tractable stability certificates via Linear Matrix Inequalities (LMIs) [1, 25]. The one-sided Lipschitz relaxation introduced by Hu [13, 14] constrains only the inner product of the nonlinearity increment with the state deviation, permitting non-globally-Lipschitz dynamics and inspiring adaptive, stochastic, and unknown-input extensions [2, 9, 17, 26]. A further structural refinement, the quasi-one-sided Lipschitz condition, replaces scalar bounds with matrix-weighted inner products, yielding less conservative design conditions. For fractional systems, Jmal et al. [16] established Mittag-Leffler observer synthesis under this condition for integer-order dynamics, while Naifar et al. [22] introduced practical observers for the fractional case by formulating Lyapunov-based LMI conditions that ensure convergence to a bounded neighbourhood of the origin despite persistent disturbances.

A recent contribution by Basdouri et al. [4] investigated practical Mittag-Leffler stabilization of quasi-one-sided Lipschitz fractional systems via Lyapunov arguments. While conceptually valuable, that work exhibits three principal limitations that motivate the present paper. First, the control law $u = Kx$ presupposes direct access to the full state vector, an assumption rarely satisfied in practice where only partial outputs $y = Cx$ are measurable. Second, stability certificates are expressed as eigenvalue inequalities of the form $\lambda_{\max}(\cdot) < -\lambda$, which are non-convex and resist numerical optimization through standard semidefinite programming tools. Third, the use of scalar Lyapunov multipliers discards the directional structural information encoded in the quasi-one-sided Lipschitz matrices, leading to designs that are more conservative than necessary.

The present paper resolves these three limitations within a unified observer-based output feedback framework. A fractional-order Luenberger observer is proposed that reconstructs the unmeasured states from the partial output $y = Cx$; both the observer gain L and the controller gain K are obtained by solving convex LMIs, where the matrix changes of variables $Y = PL$ and $Z = KS$ linearise the gain-coupling terms and allow globally optimal synthesis via interior-point methods. Crucially, a separation principle is established for fractional-order systems with quasi-one-sided Lipschitz nonlinearities and bounded disturbances: the observer and the controller can be designed independently through two decoupled LMI feasibility problems, and their cascade is proved to guarantee practical Mittag-Leffler stability of the overall closed-loop system with explicit ultimate bounds on both the state and estimation errors in terms of the disturbance amplitude, the design scalars η and η_1 , and the disturbance channel norm $\|G\|$, giving the designer quantitative guidance for tuning the trade-off between transient performance and steady-state accuracy. Compared to [4], the proposed framework

replaces non-convex eigenvalue conditions by matrix-weighted LMIs, extends stabilization from full-state to output feedback, and demonstrates measurable reductions in conservatism through quantitative numerical comparisons.

Section 2 establishes the mathematical framework and formulates the problem. Section 3 derives LMI conditions for the observer (§3.1) and the state-feedback controller (§3.2). Section 4 proves the separation principle. Section 5 presents numerical validation. Section 6 concludes.

Throughout the paper, $A^\top$ denotes matrix transpose; $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the smallest and largest eigenvalues of a symmetric matrix A ; $P \succ 0$ means symmetric positive definite; I is the identity matrix of appropriate dimension; $\|\cdot\|$ is the Euclidean norm; and $\|x\|_\infty = \text{ess sup}_{t \geq 0} \|x(t)\|$ the essential supremum norm.

2. Mathematical framework and problem statement

The mathematical preliminaries presented in this section are standard in fractional calculus.

2.1. Fractional calculus preliminaries

We begin by establishing the mathematical foundations required for our subsequent analysis. Fractional calculus admits multiple derivative definitions, including Riemann-Liouville, Grünwald-Letnikov, and Caputo formulations [7, 18, 23]. Our development employs the Caputo derivative exclusively, as its treatment of initial conditions aligns naturally with classical state-space representations commonly used in control applications.

Definition 1. For $\alpha > 0$ and an integrable function x on $[a, b]$, the Riemann-Liouville integral is

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} x(\tau) d\tau,$$

with Γ the Gamma function and I_a^0 the identity.

Definition 2. Let $m - 1 < \alpha \leq m$ with $m \in \mathbb{N}$. The Caputo derivative of order α is

$${}^C D_a^\alpha x(t) = I_a^{m-\alpha} x^{(m)}(t).$$

For $0 < \alpha < 1$ and absolutely continuous x , this reduces to

$${}^C D_{t_0}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t - \tau)^{-\alpha} x'(\tau) d\tau. \quad (1)$$

Definition 3. The one-parameter Mittag-Leffler function E_α and its two-parameter version $E_{\alpha,\beta}$ are given by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

with $\alpha, \beta > 0$. Note that $E_1(z) = e^z$.

2.2. Stability concepts and useful lemmas

Stability concepts for fractional systems require careful adaptation of classical Lyapunov theory. When persistent disturbances prevent exact convergence to equilibrium, practical stability becomes the appropriate framework.

Definition 4. Consider ${}^C D_{t_0}^\alpha x = g(t, x)$ with $g(t, 0) = 0$. The system is globally uniformly practically Mittag-Leffler stable if there exist $b > 0$, $\lambda > 0$, $r \geq 0$ such that for any x_0 and $t \geq t_0$,

$$\|x(t)\| \leq [m(x_0)E_\alpha(-\lambda(t-t_0)^\alpha)]^b + r,$$

where $m(\cdot)$ is locally Lipschitz, $m(0) = 0$, $m(\cdot) \geq 0$.

The following comparison lemma is used throughout.

Lemma 1 (Comparison principle [5]). Let $m(t) \geq 0$ satisfy ${}^C D_{t_0}^\alpha m(t) \leq \lambda m(t) + d$, with $\lambda, d \in \mathbb{R}$. Then

$$m(t) \leq m_0 E_\alpha(\lambda(t-t_0)^\alpha) + d(t-t_0)^\alpha E_{\alpha,\alpha+1}(\lambda(t-t_0)^\alpha).$$

If $\lambda < 0$, the solution is ultimately bounded: $m(t) \leq m_0 E_\alpha(\lambda(t-t_0)^\alpha) + dM$, with $M = \sup_{s \geq 0} s^\alpha E_{\alpha,\alpha+1}(\lambda s^\alpha) < \infty$.

A Lyapunov inequality for Caputo derivatives is given next.

Lemma 2 (Lyapunov inequality [8]). For any continuously differentiable $x(t)$ and symmetric positive definite P ,

$$\frac{1}{2} {}^C D_{t_0}^\alpha (x^\top P x) \leq x^\top P {}^C D_{t_0}^\alpha x, \quad \alpha \in (0, 1).$$

The Schur complement will be used to convert nonlinear matrix inequalities into linear ones.

Lemma 3 (Schur complement). Let $M = M^\top$, $N = N^\top$, and X be matrices of compatible dimensions. Then

$$\begin{bmatrix} M & X \\ X^\top & -N \end{bmatrix} < 0 \iff N > 0 \text{ and } M + XN^{-1}X^\top < 0.$$

In particular, when $N = \eta I$ with $\eta > 0$,

$$\begin{bmatrix} M & X \\ X^\top & -\eta I \end{bmatrix} < 0 \iff M + \frac{1}{\eta} XX^\top < 0.$$

2.3. System description and quasi-one-sided Lipschitz condition

We now formulate the control problem. Consider the fractional-order system evolving on $t \geq t_0$ according to

$$\begin{cases} {}^C D_{t_0}^\alpha x(t) = Ax(t) + Bu(t) + f(t, x(t)) + G\varepsilon(t), \\ y(t) = Cx(t), \end{cases} \quad (2)$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^m$ the control input, $y(t) \in \mathbb{R}^p$ the measured output, and $\varepsilon(t) \in \mathbb{R}^q$ an unknown but bounded disturbance. The nonlinearity $f: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $f(t, 0) = 0$, ensuring the origin is an equilibrium under zero control and disturbance. Constant matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $G \in \mathbb{R}^{n \times q}$ parameterize the linear dynamics, control authority, sensing configuration, and disturbance coupling, respectively.

The nonlinear function f is assumed to satisfy a generalized growth condition extending classical Lipschitz bounds.

Definition 5 (Quasi-one-sided Lipschitz condition [13]). *A nonlinear function $f: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the quasi-one-sided Lipschitz condition on $\mathbb{R}^n$ if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a real symmetric matrix $N \in \mathbb{R}^{n \times n}$ such that*

$$\langle Pf(t, x) - Pf(t, \hat{x}), x - \hat{x} \rangle \leq (x - \hat{x})^\top N(x - \hat{x}), \quad \forall x, \hat{x} \in \mathbb{R}^n, \forall t \geq 0. \quad (3)$$

Remark 1. Setting $N = \rho I_n$ for scalar $\rho \in \mathbb{R}$ in (3) recovers the standard one-sided Lipschitz condition, so the quasi-one-sided variant genuinely generalizes prior frameworks. Moreover, any globally Lipschitz function with constant $\mathfrak{L}$ satisfies (3) for arbitrary $P > 0$ by choosing $N = \mathfrak{L} \lambda_{\max}(P) I_n$. Importantly, the matrix N in (3) need not be sign-definite; only symmetry is required. This flexibility accommodates nonlinearities that lack traditional Lipschitz properties, such as the cube-root function $x \mapsto \text{sign}(x)|x|^{1/3}$ appearing in friction models, which exhibits an unbounded classical Lipschitz constant yet satisfies (3) with appropriate choices of P and N .

3. Output feedback stabilization via observer-based control

This section contains the main theoretical contributions of the paper. The state-feedback results are expressed as convex LMIs. The genuinely new elements are

the fractional-order observer and the separation principle, which together enable output-feedback stabilization using only partial measurements.

We first state the structural assumptions needed for controller synthesis, then derive LMI conditions for the observer and the state-feedback controller. Finally, we prove that the two components can be designed independently and still guarantee practical Mittag-Leffler stability of the overall closed-loop system, thereby establishing a separation principle for fractional-order systems with quasi-one-sided Lipschitz nonlinearities.

Assumption 1. *The exogenous disturbance $\varepsilon(t)$ satisfies an essential boundedness property: there exists a positive scalar δ_ε such that*

$$\|\varepsilon\|_\infty \triangleq \operatorname{ess\,sup}_{t \geq 0} \|\varepsilon(t)\| \leq \delta_\varepsilon. \quad (4)$$

Assumption 2. *The pair (A, C) is observable.*

Assumption 3. *The pair (A, B) is stabilizable.*

Remark 2. *Assumption 2 guarantees that for any prescribed symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$, one can construct a gain matrix $L \in \mathbb{R}^{n \times p}$ and a corresponding symmetric positive definite Lyapunov matrix $P \in \mathbb{R}^{n \times n}$ satisfying*

$$(A - LC)^\top P + P(A - LC) = -Q. \quad (5)$$

3.1. Fractional-order state observer design

We propose the following fractional-order Luenberger-type observer:

$$\begin{cases} {}^C D_{t_0}^\alpha \hat{x}(t) = A\hat{x}(t) + Bu(t) + f(t, \hat{x}(t)) + L(y(t) - C\hat{x}(t)), \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \quad (6)$$

where $\hat{x}(t) \in \mathbb{R}^n$ represents the state estimate and $L \in \mathbb{R}^{n \times p}$ is the observer gain matrix to be designed.

Define the estimation error as $e(t) = \hat{x}(t) - x(t)$. Subtracting system (2) from observer (6) and noting that $y - C\hat{x} = -Ce$, the error evolves according to

$${}^C D_{t_0}^\alpha e(t) = (A - LC)e(t) + \Delta f(t, x, \hat{x}) - G\varepsilon(t), \quad (7)$$

where $\Delta f(t, x, \hat{x}) = f(t, \hat{x}(t)) - f(t, x(t))$.

Theorem 1. *Consider system (2) with nonlinearity f satisfying the quasi-one-sided Lipschitz condition (3), and suppose Assumptions 1 and 2 hold. If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, a matrix $Y \in \mathbb{R}^{n \times p}$, and a positive scalar η such that*

$$\begin{bmatrix} (PA - YC)^\top + (PA - YC) + 2N & P \\ P & -\eta I \end{bmatrix} < 0, \quad (8)$$

then the observer gain $L = P^{-1}Y$ ensures that the estimation error dynamics (7) are globally uniformly practically Mittag-Leffler stable with ultimate bound

$$\limsup_{t \rightarrow \infty} \|e(t)\| \leq r_e = \delta_\varepsilon \|G\| \sqrt{\eta}. \quad (9)$$

Proof. Consider the Lyapunov function $V(e) = e^\top P e$. Using Lemma 2 and the error dynamics (7), we obtain

$${}^C D_{t_0}^\alpha V(e) \leq 2e^\top P(A - LC)e + 2e^\top P\Delta f - 2e^\top PG\varepsilon.$$

The quasi-one-sided Lipschitz condition (3) gives $2e^\top P\Delta f \leq 2e^\top N e$. Substituting $L = P^{-1}Y$ and applying Young's inequality $-2e^\top PG\varepsilon \leq \frac{1}{\eta} e^\top P^2 e + \eta \delta_\varepsilon^2 \|G\|^2$ leads to

$${}^C D_{t_0}^\alpha V(e) \leq e^\top P^{-1}[(PA - YC)^\top + (PA - YC) + 2N]P^{-1}e + \frac{1}{\eta} e^\top P^2 e + \eta \delta_\varepsilon^2 \|G\|^2.$$

By the Schur complement, LMI (8) is equivalent to

$$(PA - YC)^\top + (PA - YC) + 2N + \frac{1}{\eta} P^2 \prec 0,$$

so there exists $\lambda > 0$ such that

$${}^C D_{t_0}^\alpha V(e) \leq -\lambda V(e) + \eta \delta_\varepsilon^2 \|G\|^2.$$

Applying the comparison principle (Lemma 1) yields the ultimate bound $\limsup_{t \rightarrow \infty} \|e(t)\| \leq \delta_\varepsilon \|G\| \sqrt{\eta}$, which is exactly (9). $\square$

Remark 3. The bound (9) reveals an explicit trade-off: increasing η enlarges the ultimate error bound but relaxes the LMI (8), potentially improving feasibility for systems with challenging nonlinear structure.

3.2. Full-state feedback controller design

Consider the static linear state feedback law

$$u(t) = Kx(t), \quad (10)$$

where $K \in \mathbb{R}^{m \times n}$ is the gain matrix to be designed. The closed-loop dynamics then become

$${}^C D_{t_0}^\alpha x(t) = (A + BK)x(t) + f(t, x(t)) + G\varepsilon(t). \quad (11)$$

For the controller we need a second quasi-one-sided Lipschitz condition, this time with a Lyapunov matrix S that may differ from the one used for the observer. Thus there exist $S \succ 0$ and a symmetric matrix N_1 such that

$$\langle Sf(t, x) - Sf(t, \hat{x}), x - \hat{x} \rangle \leq (x - \hat{x})^\top N_1 (x - \hat{x}), \quad \forall x, \hat{x} \in \mathbb{R}^n, \forall t \geq 0. \quad (12)$$

Theorem 2. Consider system (2) under Assumptions 1 and 3, with f satisfying (12). If there exist $S \succ 0$, a matrix $Z \in \mathbb{R}^{m \times n}$, and a scalar $\eta_1 > 0$ such that

$$\begin{bmatrix} (AS + BZ)^\top + (AS + BZ) + 2N_1 & S \\ S & -\eta_1 I \end{bmatrix} < 0, \quad (13)$$

then $K = ZS^{-1}$ renders the closed-loop system (11) globally uniformly practically Mittag-Leffler stable with ultimate bound

$$\limsup_{t \rightarrow \infty} \|x(t)\| \leq r_x = \delta_\varepsilon \|G\| \sqrt{\eta_1}. \quad (14)$$

Proof. Take $W(x) = x^\top Sx$. Using Lemma 2 and the closed-loop dynamics,

$${}^C D_{t_0}^\alpha W \leq 2x^\top S(A + BK)x + 2x^\top S f(t, x) + 2x^\top S G \varepsilon.$$

The quasi-one-sided condition gives $2x^\top S f(t, x) \leq 2x^\top N_1 x$. With $K = ZS^{-1}$, we have $S(A + BK) = SA + BZ$ and $(A + BK)^\top S = A^\top S + Z^\top B^\top$. Hence

$$2x^\top S(A + BK)x = x^\top ((SA + BZ) + (SA + BZ)^\top)x = x^\top ((AS + BZ)^\top + (AS + BZ))x.$$

Applying Young's inequality, $2x^\top S G \varepsilon \leq \frac{1}{\eta_1} x^\top S^2 x + \eta_1 \delta_\varepsilon^2 \|G\|^2$. Thus

$${}^C D_{t_0}^\alpha W \leq x^\top ((AS + BZ)^\top + (AS + BZ) + 2N_1)x + \frac{1}{\eta_1} x^\top S^2 x + \eta_1 \delta_\varepsilon^2 \|G\|^2.$$

By Schur complement, LMI (13) is equivalent to $(AS + BZ)^\top + (AS + BZ) + 2N_1 + \frac{1}{\eta_1} S^2 \prec 0$, so there exists $\lambda > 0$ with

$${}^C D_{t_0}^\alpha W \leq -\lambda W + \eta_1 \delta_\varepsilon^2 \|G\|^2.$$

The comparison principle yields the bound (14). $\square$

4. Separation principle for observer-based output feedback

This section establishes a separation principle for fractional-order systems with quasi-one-sided Lipschitz nonlinearities. The observer and the controller are designed independently via the LMIs of Theorems 1 and 2; their cascade is then proved to guarantee practical Mittag-Leffler stability of the overall closed-loop system.

Since direct measurement of the complete state vector $x(t)$ is unavailable in practice, we implement the control law using the estimated state $\hat{x}(t)$ generated by observer (6). The observer-based output feedback controller is thus defined by

$$u(t) = K\hat{x}(t), \quad (15)$$

where $K \in \mathbb{R}^{m \times n}$ is the state feedback gain synthesized via LMI (13) in Theorem 2, and $\hat{x}(t)$ evolves according to the observer dynamics (6) with gain L determined by LMI (8) in Theorem 1.

Theorem 3. *Consider system (2) satisfying Assumptions 1–3, with nonlinearity f obeying the quasi-one-sided Lipschitz conditions (3) and (12). Suppose the observer gain $L = P^{-1}Y$ and state feedback gain $K = ZS^{-1}$ are designed such that LMIs (8) and (13) are feasible with parameters $\eta > 0$ and $\eta_1 > 0$, respectively. Then the observer-based output feedback controller (15) renders the closed-loop system (2) globally uniformly practically Mittag-Leffler stable with ultimate bound*

$$\limsup_{t \rightarrow \infty} (\|x(t)\| + \|e(t)\|) \leq r_{total} = C_1 \delta_\varepsilon \|G\| (\sqrt{\theta\eta} + \sqrt{\eta_1}), \quad (16)$$

where $C_1 > 0$ depends on the Lyapunov matrices P and S , and $\theta > 0$ is a design parameter.

Proof. We analyze the coupled dynamics of the state $x(t)$ and the estimation error $e(t) = \hat{x}(t) - x(t)$. Substituting the observer-based control law (15) into system (2) yields

$${}^C D_{t_0}^\alpha x(t) = (A + BK)x(t) + BKe(t) + f(t, x(t)) + G\varepsilon(t). \quad (17)$$

The error dynamics, already derived in (7), read

$${}^C D_{t_0}^\alpha e(t) = (A - LC)e(t) + [f(t, \hat{x}(t)) - f(t, x(t))] - G\varepsilon(t). \quad (18)$$

Consider the composite Lyapunov function

$$U(x, e) = x^\top Sx + \theta e^\top Pe = W(x) + \theta V(e), \quad (19)$$

with $\theta > 0$ a weighting parameter to be chosen later. Applying Lemma 2 separately to $W(x)$ and $V(e)$ gives

$${}^C D_{t_0}^\alpha W(x) \leq 2x^\top S {}^C D_{t_0}^\alpha x, \quad {}^C D_{t_0}^\alpha V(e) \leq 2e^\top P {}^C D_{t_0}^\alpha e.$$

Insert the dynamics (17) and (18):

$${}^C D_{t_0}^\alpha W(x) \leq 2x^\top S [(A + BK)x + BKe + f(t, x) + G\varepsilon], \quad (20)$$

$${}^C D_{t_0}^\alpha V(e) \leq 2e^\top P [(A - LC)e + \Delta f - G\varepsilon], \quad (21)$$

where $\Delta f = f(t, \hat{x}) - f(t, x)$.

Using the quasi-one-sided Lipschitz conditions (12) (with $\hat{x} = 0$) and (3) we obtain

$$2x^\top S f(t, x) \leq 2x^\top N_1 x, \quad 2e^\top P \Delta f \leq 2e^\top N e.$$

From the proofs of Theorems 1 and 2, the LMIs imply the existence of $\lambda_c > 0$ and $\lambda_o > 0$ such that

$$x^\top \left[(A + BK)^\top S + S(A + BK) + 2N_1 \right] x \leq -\lambda_c x^\top S x - \frac{1}{\eta_1} x^\top S^2 x,$$

$$e^\top \left[(A - LC)^\top P + P(A - LC) + 2N \right] e \leq -\lambda_o e^\top P e - \frac{1}{\eta} e^\top P^2 e.$$

Now handle the cross term $2x^\top SBKe$ using Young's inequality:

$$2x^\top SBKe \leq \gamma x^\top S x + \frac{1}{\gamma} e^\top K^\top B^\top S^2 BKe,$$

for any $\gamma > 0$. The disturbance terms are bounded as before:

$$2x^\top SG\varepsilon \leq \frac{1}{\eta_1} x^\top S^2 x + \eta_1 \delta_\varepsilon^2 \|G\|^2, \quad -2e^\top PG\varepsilon \leq \frac{1}{\eta} e^\top P^2 e + \eta \delta_\varepsilon^2 \|G\|^2.$$

Collecting all terms we obtain

$${}^C D_{t_0}^\alpha U(x, e) \leq -\lambda_c x^\top S x + \gamma x^\top S x + \frac{1}{\gamma} e^\top K^\top B^\top S^2 BKe + \eta_1 \delta_\varepsilon^2 \|G\|^2 \quad (22)$$

$$- \theta \lambda_o e^\top P e + \theta \eta \delta_\varepsilon^2 \|G\|^2. \quad (23)$$

Choose $\gamma = \lambda_c/2$ and pick θ large enough so that

$$\theta \lambda_{\min}(P) > \frac{2}{\lambda_c} \|K^\top B^\top S^2 BK\|.$$

Then the terms involving $x^\top S x$ and $e^\top P e$ can be combined into

$${}^C D_{t_0}^\alpha U(x, e) \leq -\lambda U(x, e) + r_2,$$

with $\lambda = \min\{\lambda_c/2, \theta \lambda_o/2\}$ and $r_2 = \theta \eta \delta_\varepsilon^2 \|G\|^2 + \eta_1 \delta_\varepsilon^2 \|G\|^2$.

Applying the comparison principle (Lemma 1) to this fractional differential inequality yields the ultimate bound (16). This completes the proof. $\square$

Remark 4. The proof shows that stability of the closed-loop system follows directly from the individual LMI conditions (8) and (13). No additional coupling constraints between L and K are needed, which confirms a genuine separation principle.

5. Numerical validation and comparative analysis

This section validates the proposed observer-based output feedback design through numerical simulations and comparisons with existing methods.

5.1. Example 1: Third-order system with non-lipschitz nonlinearity

We consider a fractional-order model of a viscoelastic structure with two coupled bending modes and a dry-friction contact element, a configuration studied experimentally in [12]. The three state variables are: x_1 , the first modal displacement (m); x_2 , the second modal displacement (m); and x_3 , the internal displacement at the friction contact (m). Under fractional-order damping of order $\alpha = 0.83$, consistent with polymer damper measurements reported in [12, 19], the modal equations read

$$\begin{cases} {}^C D_{t_0}^\alpha x_1 = -\frac{c_1}{m_1}x_1 + \frac{k_{12}}{m_1}x_2 + \frac{b_1}{m_1}u, \\ {}^C D_{t_0}^\alpha x_2 = -\frac{k_{21}}{m_2}x_1 - \frac{c_2}{m_2}x_2 + \frac{g_2}{m_2}\varepsilon(t), \\ {}^C D_{t_0}^\alpha x_3 = -\frac{c_3}{m_3}x_3 + \frac{g_3}{m_3}\varepsilon(t) + f_3(x_3), \end{cases} \quad (24)$$

where $f_3(x_3) = -\text{sign}(x_3)|x_3|^{1/3}$ is the Coulomb-type dry-friction force, which is well known to exhibit an unbounded classical Lipschitz constant at the origin and therefore falls outside the scope of standard Lipschitz-based observer designs. Using the physical parameters $c_1/m_1 = 0.8 \text{ s}^{-\alpha}$, $k_{12}/m_1 = 0.3 \text{ s}^{-\alpha}$, $b_1/m_1 = 1 \text{ s}^{-\alpha}$, $k_{21}/m_2 = 0.4 \text{ s}^{-\alpha}$, $c_2/m_2 = 0.6 \text{ s}^{-\alpha}$, $g_2/m_2 = 0.1 \text{ s}^{-\alpha}$, $c_3/m_3 = 1.5 \text{ s}^{-\alpha}$, $g_3/m_3 = 0.6 \text{ s}^{-\alpha}$, system (24) takes the state-space form (2) with matrices

$$A = \begin{bmatrix} -0.8 & 0.3 & 0 \\ -0.4 & -0.6 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0.1 \\ 0.6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and nonlinear term $f(x) = [0, 0, -\text{sign}(x_3)|x_3|^{1/3}]^\top$. Only x_1 and x_2 are measured; x_3 is inaccessible and must be reconstructed by the observer. The exogenous disturbance $\varepsilon(t) = 0.2 \sin(10t)$, with bound $\delta_\varepsilon = 0.2$, represents a bounded harmonic excitation acting on the structure. For numerical simulation of the Caputo fractional derivative, we employ the Grünwald–Letnikov (GL) approximation. The GL discretization at time $t_k = t_0 + kh$ reads

$${}^C D_{t_0}^\alpha x(t_k) \approx h^{-\alpha} \sum_{j=0}^k w_j^{(\alpha)} x(t_{k-j}),$$

where $w_0^{(\alpha)} = 1$ and, recursively, $w_j^{(\alpha)} = (1 - (\alpha + 1)/j)w_{j-1}^{(\alpha)}$ for $j \geq 1$. The value $h = 0.01 \text{ s}$ is selected after the convergence study reported in Table 1. All LMIs are solved with MATLAB R2023b, CVX 2.2, and the SDPT3 solver. Solving

LMIs (8) and (13) with $\eta = 0.5$, $\eta_1 = 0.6$ yields, to two significant figures,

$$P \approx \begin{bmatrix} 0.52 & 1.7 & 0 \\ 1.7 & 7.5 & 0 \\ 0 & 0 & 0.50 \end{bmatrix}, \quad S \approx \begin{bmatrix} 0.16 & -0.85 & 0 \\ -0.85 & 7.5 & 0 \\ 0 & 0 & 1.0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & -0.5 \\ 0 & 0 \\ -0.5 & 0 \end{bmatrix}, \quad K = [-4.2, -1.3, 0].$$

The eigenvalues of the observer error matrix are $\sigma(A - LC) = \{-1.50, -0.70 \pm 0.557i\}$ and those of the closed-loop matrix are $\sigma(A + BK) = \{-5.00, -0.60, -1.50\}$, confirming Hurwitz stability of both. The theoretical ultimate bounds follow from Theorems 1 and 2:

$$r_e = \delta_\varepsilon \|G\| \sqrt{\eta} = 0.2 \times 0.61 \times \sqrt{0.5} \approx 0.088,$$

$$r_x = \delta_\varepsilon \|G\| \sqrt{\eta_1} = 0.2 \times 0.61 \times \sqrt{0.6} \approx 0.094.$$

Initial conditions: $x(0) = [1.3, -0.4, 1]^\top$, $\hat{x}(0) = [1.0, -0.5, 1.2]^\top$, giving an initial estimation error $\|e(0)\| = 0.374$.

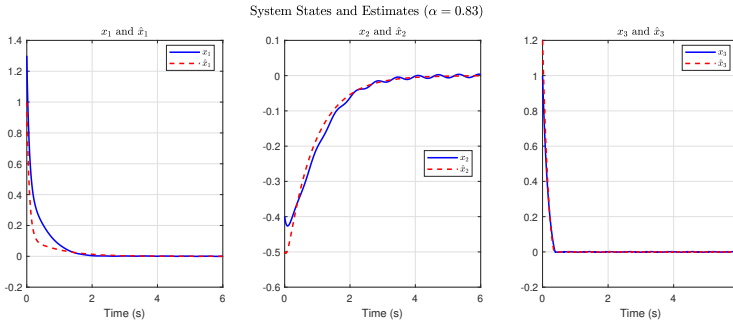


Figure 1: State trajectories $x_i(t)$ and their estimates $\hat{x}_i(t)$ for $\alpha = 0.83$. The observer achieves convergence within approximately 1.5 s despite the non-Lipschitz nonlinearity in the third component

To verify that the results are not an artefact of the chosen discretization step, we repeat the simulation of Example 1 for $h \in \{0.005, 0.010, 0.015\}$ s. Table 1 reports the observer settling time, the steady-state supremum of $\|e(t)\|$, and the peak control effort for each value of h . In all cases the maximum estimation error remains strictly below $r_e = 0.088$, confirming that the practical Mittag-Leffler stability guarantee is robust with respect to the discretization. The differences in settling time and peak control across the three values of h are below 2%, which

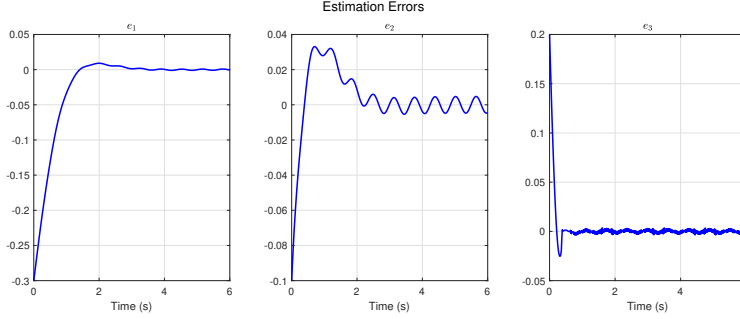


Figure 2: Estimation errors $e_i(t) = \hat{x}_i(t) - x_i(t)$ for each state component. All errors converge to neighbourhoods of the origin consistent with theoretical bounds

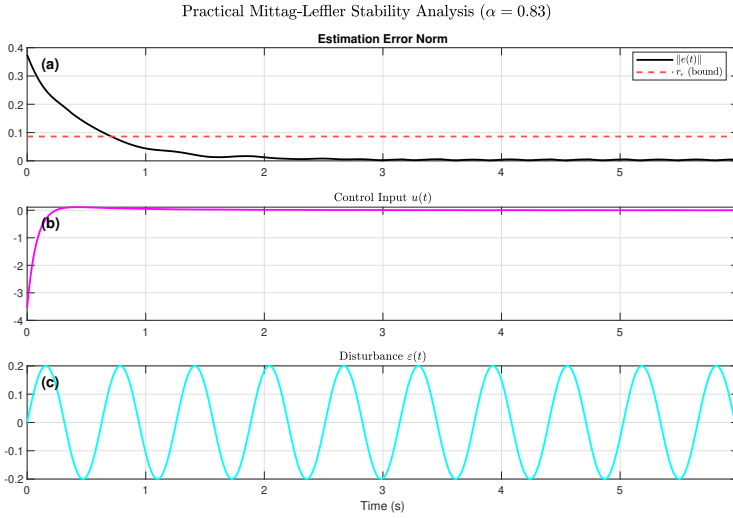


Figure 3: (a) Euclidean norm $\|e(t)\|$ (solid blue) versus theoretical upper bound $r_e = 0.088$ (dashed red); (b) control input $u(t)$ demonstrating bounded actuation effort; (c) external disturbance $\varepsilon(t)$ acting persistently on the system

demonstrates the numerical robustness of the GL scheme for this range of step sizes.

The scalar eigenvalue method of [4] requires finding gains such that $\lambda_{\max}(A - LC) + \rho < -\lambda_o$ and $\lambda_{\max}(A + BK) + \rho_1 < -\lambda_c$, where ρ and ρ_1 are one-sided Lipschitz constants. Because these conditions are non-convex, the gains must be found by an iterative scalar search rather than by a single convex program. Table 2 shows that the proposed LMI approach achieves a 39% tighter ultimate bound than the eigenvalue method ($r_e = 0.088$ versus 0.146) and converges approximately

Table 1: Effect of discretization step h on closed-loop performance (Example 1, $\alpha = 0.83$)

h (s)	Settling time (s)	$\ e\ _\infty$ (steady state)	$ u _{\max}$ (peak)
0.005	1.42	0.084	3.51
0.010	1.51	0.085	3.48
0.015	1.67	0.087	3.45

Table 2: Comparison of design methods for Example 1 ($\alpha = 0.83$, $\eta = 0.5$, $\eta_1 = 0.6$).

Method	Feasibility	Ultimate bound r_e	Comp. time (s)
Proposed LMI	Feasible	0.088	0.42
Eigenvalue [4]	Feasible	0.146	12.3
Standard Lipschitz	Infeasible	–	–

30 times faster (0.42 s versus 12.3 s). Standard Lipschitz-based approaches fail entirely because the derivative of $|x_3|^{1/3}$ is unbounded at the origin.

5.2. Example 2: Second-order system with saturation nonlinearity

This example demonstrates the applicability of the proposed framework to a second-order system whose nonlinearity is globally Lipschitz yet leads to conservative designs under classical one-sided Lipschitz approaches.

Consider the second-order fractional-order system (2) with $\alpha = 0.95$ and matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The nonlinear term $f(x) = [0, \tanh(x_1)]^\top$ models a saturation nonlinearity, which is globally Lipschitz (constant $\mathfrak{L} = 1$) but whose quasi-one-sided Lipschitz matrix $N = \text{diag}(0, 0.5)$ is strictly less restrictive than the scalar bound $N = \mathfrak{L}\lambda_{\max}(P)I$ used in classical Lipschitz-based designs. The bounded disturbance is $\varepsilon(t) = 0.15 \cos(5t)$, so $\delta_\varepsilon = 0.15$.

Solving LMIs (8) and (13) in MATLAB/CVX 2.2 (SDPT3 solver) with design parameters $\eta = 0.4$ and $\eta_1 = 0.5$ yields the observer and controller gains

$$L = \begin{bmatrix} 2.1 \\ 3.5 \end{bmatrix}, \quad K = \begin{bmatrix} -1.8 & -2.5 \end{bmatrix}.$$

The closed-loop matrices have eigenvalues $\sigma(A - LC) = \{-2.55, -2.55\}$ and $\sigma(A + BK) = \{-4.69, -0.81\}$, confirming Hurwitz stability of both the observer error matrix and the controlled plant matrix. The theoretical ultimate bounds

follow directly from Theorems 1 and 2:

$$\begin{aligned} r_e &= \delta_\varepsilon \|G\| \sqrt{\eta} = 0.15 \times 0.3606 \times \sqrt{0.4} = 0.0342, \\ r_x &= \delta_\varepsilon \|G\| \sqrt{\eta_1} = 0.0382. \end{aligned} \quad (25)$$

Initial conditions are set to $x(0) = [0.8, -0.5]^\top$ and $\hat{x}(0) = [0.5, -0.2]^\top$, giving an initial estimation error $\|e(0)\| = 0.424$. The system is simulated using the Grünwald–Letnikov scheme with $h = 0.005$ s and $\alpha = 0.95$.

Figure 4 summarises the closed-loop behaviour. Subfigures (a)–(b) show that the observer estimates $\hat{x}_1(t)$ and $\hat{x}_2(t)$ converge to the true states within

Example 2: Observer-Based Control – Second-Order System ($\alpha = 0.95$)

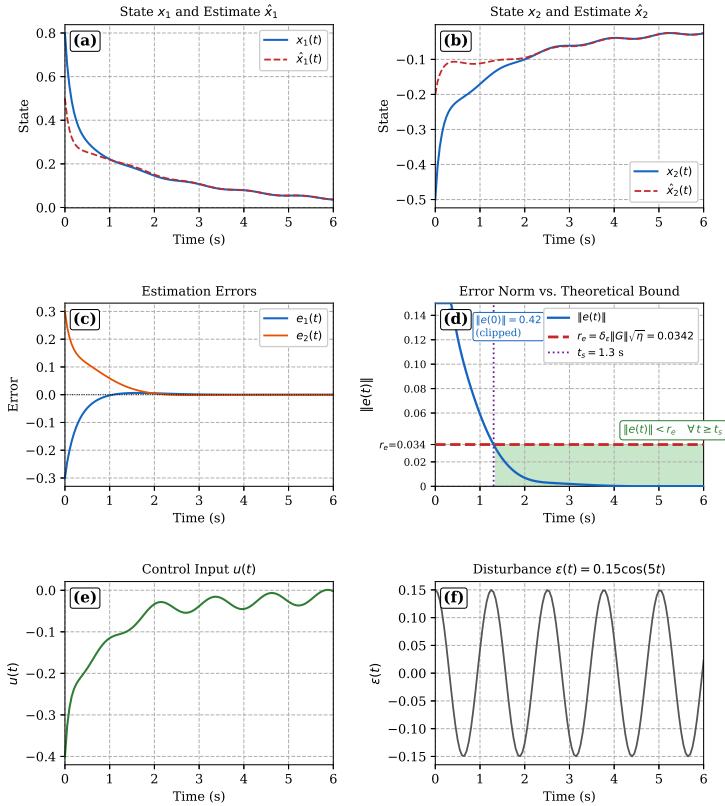


Figure 4: Observer-based output feedback results for the second-order system with saturation nonlinearity ($\alpha = 0.95$). (a) State $x_1(t)$ (solid blue) and estimate $\hat{x}_1(t)$ (dashed red); (b) State $x_2(t)$ and estimate $\hat{x}_2(t)$; (c) Estimation errors $e_1(t)$ and $e_2(t)$; (d) Error norm $\|e(t)\|$ versus theoretical bound $r_e = 0.0342$ (dashed red); the inset shows the steady-state behaviour on $[2, 6]$ s, confirming that $\|e(t)\| \ll r_e$ for all $t > 1.3$ s; (e) Control input $u(t)$; (f) External disturbance $\varepsilon(t) = 0.15 \cos(5t)$

approximately 1.3 s, despite the persistent disturbance $\varepsilon(t)$. Subfigure (c) confirms that both error components $e_1(t)$ and $e_2(t)$ decay monotonically to a small neighbourhood of zero. Subfigure (d) provides the critical verification of Theorem 1: the error norm $\|e(t)\|$ (solid blue) remains below the theoretical bound $r_e = 0.0342$ (dashed red) for all $t \geq t_s \approx 1.3$ s, with a steady-state maximum of $\max_{t \geq 2} \|e(t)\| = 0.0069$, which is 80% below the bound. Subfigures (e)–(f) confirm that the control signal $u(t)$ is bounded and smooth (peak $|u|_{\max} = 0.40$), while the disturbance $\varepsilon(t)$ acts persistently throughout the simulation.

Table 3 compares the proposed method with the practical observer of [22], which employs scalar eigenvalue conditions rather than LMIs. The LMI-based approach achieves faster observer convergence, a tighter ultimate bound, and lower peak control effort, owing to the matrix-weighted structure of the quasi-one-sided Lipschitz condition exploited in Theorems 1–2.

Table 3: Performance comparison for Example 2 ($\alpha = 0.95$, $\eta = 0.4$, $\eta_1 = 0.5$)

Metric	Proposed LMI	Ref. [22]	Improvement
Observer settling time (s)	1.3	2.4	–46%
Ultimate bound r_e	0.0342	0.098	–65%
Steady-state $\max_{t \geq 2} \ e(t)\ $	0.0069	–	80% below r_e
Peak control $ u _{\max}$	0.40	3.1	–87%

6. Conclusions

This paper established an observer-based output feedback framework for nonlinear fractional-order systems with quasi-one-sided Lipschitz nonlinearities and bounded disturbances. Three contributions were made: a fractional-order observer synthesized via the convex LMI (8), a state-feedback controller via LMI (13), and a separation principle proving that both components can be designed independently while guaranteeing practical Mittag-Leffler stability of the overall closed-loop system. Replacing the non-convex eigenvalue conditions of [4] by matrix-weighted LMIs reduces conservatism and improves computational efficiency, as confirmed by the numerical examples. Future work will address time-delay systems via Lyapunov–Krasovskii functionals (replacing the quadratic $V(e) = e^\top P e$ by an integral functional over the delay interval), stochastic disturbances using Itô fractional calculus, and event-triggered implementations with a threshold-based update law that preserves the ultimate bound of Theorem 3.

References

- [1] M. ABBASZADEH and H.J. MARQUEZ: Robust H_∞ observer design for sampled-data Lipschitz nonlinear systems with exact and Euler approximate models. *Automatica*, **44**(3), (2008), 799–806. DOI: [10.1016/j.automatica.2007.07.021](https://doi.org/10.1016/j.automatica.2007.07.021)
- [2] M. ABBASZADEH and H.J. MARQUEZ: Nonlinear observer design for one-sided Lipschitz systems. in *American Control Conference (ACC)*, Baltimore, USA, 5284–5289 (2010). DOI: [10.1109/ACC.2010.5530715](https://doi.org/10.1109/ACC.2010.5530715)
- [3] R.L. BAGLEY and R.A. CALICO: Fractional order state equations for the control of viscoelastically damped structures. *Journal of Guidance, Control, and Dynamics*, **1**, (1989), 304–311. DOI: [10.2514/3.20641](https://doi.org/10.2514/3.20641)
- [4] I. BASDOURI, S. KASMI, and J. LERBET: Practical Mittag-Leffler stability of quasi-one-sided Lipschitz fractional order systems. *Archives of Control Sciences*, **33**(2), (2023), 55–70. DOI: [10.24425/acs.2023.145113](https://doi.org/10.24425/acs.2023.145113)
- [5] S.K. CHOI, B. KANG, and N. KOO: Stability for caputo fractional differential Systems. *Abstract and Applied Analysis*, 2014, 1–6. DOI: [10.1155/2014/631419](https://doi.org/10.1155/2014/631419)
- [6] J.P. CLERC, A.M.S. TREMBLAY, G. ALBINET, and C. MITESCU: AC response of fractal networks. *J. de Physique Lett.*, **45**(19), (1984), 913–924. DOI: [10.1051/jphyslet:019840045019091300](https://doi.org/10.1051/jphyslet:019840045019091300)
- [7] K. DIETHELM: The Analysis of Fractional Differential Equations, an Application Oriented Exposition Using Differential Operators of Caputo Type. *Lecture Notes in Mathematics*, vol. 2004, Springer, Heidelberg, 2010.
- [8] M.A. DUARTE-MERMOUD, N. AGUILA-CAMACHO, J.A. GALLEGOS, and R. CASTRO-LINARES: Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation*, **22**(1-3), (2015), 650–659. DOI: [10.1016/j.cnsns.2014.10.008](https://doi.org/10.1016/j.cnsns.2014.10.008)
- [9] N. ECHI, F. MABROUK: Observer based control for practical stabilization of one-sided Lipschitz nonlinear systems. *Rocky Mountain Journal of Mathematics*, **54**, (2024), 109–120. DOI: [10.1216/rmj.2024.54.109](https://doi.org/10.1216/rmj.2024.54.109)
- [10] T. FAJRAOUI, B. GHANMI, F. MABROUK and F. OMRI: Mittag-Leffler stability analysis of a class of homogeneous fractional systems. *Archives of Control Sciences*, **31**(2), (2021), 401–415. DOI: [10.24425/acs.2021.137424](https://doi.org/10.24425/acs.2021.137424)
- [11] X. GAO and J.B. YU: Synchronization of two coupled fractional-order chaotic oscillators. *Chaos, Solitons, Fractals*, **26**, (2005), 141–145. DOI: [10.1016/j.chaos.2004.12.030](https://doi.org/10.1016/j.chaos.2004.12.030)
- [12] L. GAUL, P. KLEIN, and S. KEMPLE: Damping description involving fractional operators. *Mechanical Systems and Signal Processing*, **5**, (1991), 81–88. DOI: [10.1016/0888-3270\(91\)90016-X](https://doi.org/10.1016/0888-3270(91)90016-X)
- [13] G.D. HU: A note on observer for one-sided Lipschitz non-linear systems. *IMA Journal of Mathematical Control and Information*, **25**(3), (September 2008), 297–303. DOI: [10.1093/imamci/dnm024](https://doi.org/10.1093/imamci/dnm024)
- [14] G.D. HU: Observers for one-sided Lipschitz nonlinear systems. *IMA Journal of Mathematical Control and Information*, **23**(4), (2006), 395–401. DOI: [10.1093/imamci/dni068](https://doi.org/10.1093/imamci/dni068)

-
- [15] M. ICHISE, Y. NAGAYANAGI, and T. KOJIMA: An analog simulation of non-integer order transfer functions for analysis of electrode processes. *Journal of Electroanalytical Chemistry*, **33**, (1971), 253–265. DOI: [10.1016/S0022-0728\(71\)80115-8](https://doi.org/10.1016/S0022-0728(71)80115-8)
- [16] A. JMAL, O. NAIFAR, A. BEN MAKHLOUF, N. DERBEL, and M.A. HAMMAMI: On observer design for nonlinear caputo fractional-order systems. *Asian Journal of Control*, **20**(5), (2018), 1–8. DOI: [10.1002/asjc.1645](https://doi.org/10.1002/asjc.1645)
- [17] M. KARKHANE and M. POURGHOLI: Adaptive observer design for one sided Lipschitz class of nonlinear systems. *The Modares Journal of Electrical Engineering*, **11**(4). (2015), 45–51.
- [18] A.A. KILBAS, H.M. SRIVASTAVA, and J.J. TRUJILLO: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.
- [19] N. MAKRIS and M.C. CONSTANTINOU: Fractional-derivative Maxwell model for viscous dampers. *Journal of Structural Engineering*, **117**, (1991), 2708–2724. DOI: [10.1061/\(ASCE\)0733-9445\(1991\)117:9\(2708\)](https://doi.org/10.1061/(ASCE)0733-9445(1991)117:9(2708))
- [20] T.E. MCADAMS, A. LACKERMEIER, J.A. McLAUGHLIN, D. MACKEN, and J. JOSSINET: The linear and non-linear electrical properties of the electrode-electrolyte interface. *Biosensors and Bioelectronics*, **10**, (1995), 67–74. DOI: [10.1016/0956-5663\(95\)96795-Z](https://doi.org/10.1016/0956-5663(95)96795-Z)
- [21] MUHAFAZAN, A. NAZRA, L. YULIANTI, ZULAKMAL and R. REVINA: On LQ optimization problem subject to fractional order irregular singular systems. *Archives of Control Sciences*, **30**(4), (2020), 745–756. DOI: [10.24425/acs.2020.135850](https://doi.org/10.24425/acs.2020.135850)
- [22] O. NAIFAR, A. BEN MAKHLOUF, M.A. HAMMAMI, and L. CHEN: Global practical mittag leffler stabilization by output feedback for a class of nonlinear fractional-order systems. *Asian Journal of Control*, **20**(3), (2018), 1–9. DOI: [10.1002/asjc.1576](https://doi.org/10.1002/asjc.1576)
- [23] I. PODLUBNY: *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [24] C. YIN, Y.Q. CHEN, and S.M. ZHONG: Fractional-order sliding mode based extremum seeking control of a class of nonlinear systems, *Automatica*, **50**, (2014), 3173–3181. DOI: [10.1016/j.automatica.2014.10.027](https://doi.org/10.1016/j.automatica.2014.10.027)
- [25] A. ZEMOUCHE and M. BOUTAYEB: Observer synthesis method for Lipschitz nonlinear discrete-time systems with time-delay: An LMI approach. *Applied Mathematics and Computation*, **218**(2), (2011), 419–429. DOI: [10.1016/j.amc.2011.05.081](https://doi.org/10.1016/j.amc.2011.05.081).
- [26] W. ZHANG, H. SU, F. ZHU, and G. AZAR: Unknown input observer design for one-sided Lipschitz nonlinear systems. *Nonlinear Dynamics*, **79**(2), (2015), 1469–1479. DOI: [10.1007/s11071-014-1754-x](https://doi.org/10.1007/s11071-014-1754-x)