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The State Space Model of the Scalar, Fractional Order, Variable Parameters Dynamic System

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Abstract. In the paper the new, discrete, variable parameters, scalar, fractional order state space model is addressed. The proposed model uses Fractional Order Backward Difference approximation with linearly time varying parameters: fractional order and quasi time constant. Elementary properties of the considered model: practical stability, accuracy and positivity are discussed. Theoretical considerations are illustrated by numerical tests. The proposed model can be applied to modelling of physical phenomena hard to describe with the use of time invariant approach.

Key words: fractional order state equation, time-variant system, FOBD approximation, practical stability, accuracy, positivity.

1. ACRONYMS

FO - Fractional Order,
C definition - Caputo definition,
GL definition - Grünwald-Letnikov definition,
FOBD - Fractional Order Backward Difference,
FOPID - Fractional Order PID controller,
RL definition - Riemann-Liouville definition,
VO - Variable Order,
VT - Variable Time Constant,
VOVT - Variable Order and Variable Time Constant.

2. INTRODUCTION

Variable parameter and variable fractional order models are expected to precisely describe a behaviour of physical systems or phenomena, for which the classic, constant parameter model is not enough. VO FOPID controllers are also required to assure better control performance as FOPID controllers with fixed orders.

The first proposal of a VO fractional derivative was developed by [1]. Their definition was based on the VO RL integral and the Marchaud derivative. Later [2] give definitions of VO derivative for RL, C and GL definitions as well as theoretical background for VO. In addition, this paper proposes also the use of Fuzzy Logic to approximate of VO.

Various approximation methods for variable fractional order systems have been known since years. For example, the time-continuous approximation of VO system using C definition is proposed by [3]. An alternative approach to VO operator, using Scarpi integral and derivative is proposed by [4].

An estimation of variable parameters fractional system using Triple Estimation algorithm is presented by [5], the analytical solution of VO differential equations has been proposed by [6]. The solution of variable fractional order state equation with distributed delays is given by [7]. In [8] stability conditions for variable fractional dynamic systems especially dedicated to chaotic systems are proposed.

State of the art in the area of VO fractional operator has been given by [9]. It covers problems from theoretical background to various applications of VO operator. Applications in special issues of mechanics (e.g. viscoelastics), transport processes and anomalous diffusion, fractals, VO filters are recalled as well as applications of VO FOPID controller and other control algorithms employing VO operator.

The papers of [10] and [11] propose the discrete approximation using GL definition and FOBD approximation as well as its application to construct variable fractional order inertial and oscillatory transfer functions. In these papers the time variable fractional order was included to FOBD approximator factors. This approach has been continued in the article [12], proposing the use of variable fractional order in the FOPID controller.

Variable parameters fractional systems with time-invariant order need to be mentioned too. Latest research in this area focuses mainly on control problems. For example [13] deals with time optimal control, [14], [15], [16] and [17] analyze the controllability and observability of these systems. The paper [18] proposes the extension of the Floquet-Lyapunov transformation to the fractional discrete-time linear systems with periodically varying parameters.

The new, VOVT inertial transfer function model is proposed in the paper [19]. This paper analyzes its main properties, but the stability conditions can only be given for specific cases. In addition, they have a form of sufficient or necessary conditions. This implies that they are not able to give full information about stability.

3. PRELIMINARIES

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A. Basics of fractional calculus

Elementary ideas from fractional calculus can be found in many books, e.g. [20], [21], [22] or [23]. Here only some definitions necessary to present of main results will be recalled. First of all the fractional-order, integro-differential operator (see e.g. [20], [24], [23]) needs to be given. Is is as follows:

Definition 1. *The elementary fractional order operator The fractional-order integro-differential operator is defined as follows:*

$$ {}_{t_s}D^{\alpha}_{t_f} f(t) = \begin{cases} \frac{d^{\alpha} f(t)}{dt^{\alpha}} & \alpha > 0 \\ f(t) & \alpha = 0 \\ \int_{t_s}^{t_f} f(\tau)(d\tau)^{\alpha} & \alpha < 0 \end{cases} . \quad (1)$$

where t_s and t_f denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation.

Next recall the Gamma Euler function given e.g. by [24]:

Definition 2. *The Gamma function*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt. \quad (2)$$

Mittag-Leffler function is a non-integer order generalization of exponential function $e^{\lambda t}$ and it plays crucial role in solution of FO state equation. The one parameter Mittag-Leffler function is defined as follows:

Definition 3. *The one parameter Mittag-Leffler function*

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}. \quad (3)$$

The two parameter Mittag-Leffler function is defined as follows:

Definition 4. *The two parameters Mittag-Leffler function*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}. \quad (4)$$

For $\beta = 1$ the two parameter function (4) turns to one parameter function (3).

The fractional-order, integro-differential operator can be described by different definitions, given by Grünvald and Letnikov (GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). In the further consideration only GL definition will be employed. It is recalled below.

The GL derivative along the time from function $g(t)$ is defined as follows [25] [26]:

Definition 5. *The Grünwald-Letnikov definition*

$$ {}_0^{GL}D^{\alpha}_t g(t) = \lim_{h \to 0} h^{-\alpha} \sum_{l=0}^{[\frac{t}{h}]} (-1)^l \binom{\alpha}{l} g(t - lh). \quad (5)$$

In (5) $\alpha \in \mathbb{R}$ is the fractional order along the time, h is the sample time, [..] is the nearest integer value, $\binom{\alpha}{l}$ is the binomial coefficient:

$$\binom{\alpha}{l} = \left\{ \begin{array}{cc} 1, & l = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-l+1)}{l!}, & l > 0 \end{array} \right\}. \quad (6)$$

The elementary, scalar input-output differential equation using elementary fractional operator (1) takes the following form:

$$T D^{\alpha}_t g(t) = -g(t) + u(t). \quad (7)$$

where T is the quasi time constant. It is the analogue of time constant for IO dynamic systems. It is expressed in $[s^{\frac{1}{\alpha}}]$ to to ensure consistency of units in a FO differential equation and $u(t)$ is the control signal.

B. The FOBD approximator and its variable order generalization

The GL definition is the limit case for $h \to 0$ and $L \to \infty$ of the FOBD, commonly employed in discrete FO calculations (see e.g. [22], p. 68).

Definition 6. *The Fractional Order Backward Difference along the time*

$$\Delta^{\alpha} g(t) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} g(t - lh). \quad (8)$$

In (8) h is the sample time. Its value is constant in the whole considered time interval. L denotes a memory length necessary to correct approximation of a non integer order operator. Unfortunately, good accuracy of approximation requires to use a long memory L which can generate problems during implementation.

It is worth to note that the use of FOBD operator turns the FO model to finite dimensional, discrete IO model. This allows to analyse it with the use of tools and methods known in Control Theory since years.

Denote coefficients $(-1)^l \binom{\alpha}{l}$ by d_l:

$$d_l = (-1)^l \binom{\alpha}{l}. \quad (9)$$

The coefficients (9) can also be computed using the following, equivalent, recursive formula given, e.g. by [25], p. 12, useful in numerical calculations:

$$\begin{aligned} d_0 &= 1, \\ d_l &= \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \quad l = 1, \ldots, L. \end{aligned} \quad (10)$$

In [27] it is proved that:

$$\sum_{l=1}^{\infty} d_l = 1-\alpha. \tag{11}$$

From (10) and (11) we obtain:

$$\sum_{l=2}^{\infty} d_l = 1. \tag{12}$$

Proof. From (10) we have:

$$d_1 = -\alpha,$$

and the whole sum (11) is equal $1-\alpha$. Omitting its 1'st factor gives 1.

□

Using (9) the operator (8) can be expressed in a shorter form:

$$\Delta^{\alpha} g(t) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} d_l g(t-lh), \tag{13}$$

C. The discrete,elementary FO differential equation.

The discrete version of the equation (7) using fractional difference operator Δ^{α} is as follows:

$$T\Delta^{\alpha} g(k) = -g(k) + u(k). \tag{14}$$

where k are discrete time instants.

The difference equation (14) has a solution determined both by an initial condition and control signal.

Next, the general form of a discrete, FO linear state space equation has the following form:

$$\begin{cases} {}_0\Delta_t^{\alpha} x(k) = Ax(k-1) + Bu(k-1) \\ y(k) = Cx(k) \end{cases}. \tag{15}$$

In (15) $\alpha \in \mathbb{R}$ is the fractional order, $x(k) \in \mathbb{R}^N$, $u(k) \in \mathbb{R}^L$, $y(k) \in \mathbb{R}^P$ are the state, control and output vectors respectively, A, B, C are the state, control and output operators, respectively, k denotes time instants.

The use of fractional operator allows to consider an initial condition as a function defined in the "past" before an initial moment. This gives an accurate description of reality, because the past behavior of dynamical systems is better expressed by a function than by a single value in one moment.

D. The Practical Stability and Final Value Theorem for discrete systems

An idea of practical stability has been proposed by [27]. It allows to associate the stability of FO system to stability of its approximation.

Definition 7. *The practical stability*
The fractional order system is practically stable iff its approximation is asymptotically stable.

Next the Final Value Theorem for discrete system needs to be recalled.

Theorem 1. *(Final Value Theorem for discrete time) Let $f(k)$ be a discrete function of time, defined in k time moments, and let $F(z)$ be its z-transform. Assume that $F(z)$:*

1. *has no poles outside the unit circle,*
2. *has maximally one pole on the unit circle: $z = 1$.*

then:

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1} (z-1)F(z). \tag{16}$$

E. Positivity.

Next definitions and conditions of internal and external positivity of the discrete FO system (15) should be recalled.

An idea and necessary and sufficient condition of the internal positivity of the discrete FO system is mentioned e.g. in [28].

Definition 8. *(The internal positivity)*
The discrete FO system (15) is called internally positive if: $x(k) \in \mathbb{R}_+^N$, $y(k) \in \mathbb{R}_+^P$, $t \geq 0$ for any initial conditions $x_0 \in \mathbb{R}_+^N$ and all inputs $u(k) \in \mathbb{R}_+^M$, where ${\mathbb{R}_+}^{N,M,P}$ is the N, M, P dimensional space of non negative real numbers.

Theorem 2. *The discrete FO system (15) is internally positive iff:*

$$A \in \mathbb{R}_+^{N\times N},\ B \in \mathbb{R}_+^{N\times M},\ C \in \mathbb{R}_+^{P\times N}. \tag{17}$$

where $\mathbb{R}_+^{N\times N}$, $\mathbb{R}_+^{N\times M}$ and $\mathbb{R}_+^{P\times N}$ are sets of matrices with non negative entires with suitable dimensions.

The external positivity of a discrete system is mentioned e.g. in [29]. Its definition deals with a behaviour of a response of a system to positive control.

Definition 9. *(The external positivity)*
The FO system (15) is called externally positive if:

$$x_0 = 0 \wedge u(k) \in \mathbb{R}_+^M \implies y(k) \in \mathbb{R}_+^P \ \ k = 0, 1, 2, ...$$

.

The internal positivity always implies the external positivity, but the reverse implication is not a true. The proving of the external positivity without internal positivity is not a trivial issue.

4. THE PROPOSED VOVT STATE SPACE MODEL

Consider the elementary FO differential equation (7).

Assume that the order α can take various values at discrete time instants:

$$\alpha = \{\alpha_0, \alpha_1, ..., \alpha_L\},\ \alpha_l > 0,\ l = 0, 1, ..., L. \tag{18}$$

The variable order generalization of the operator (8) using various orders (18) has been proposed in [30] and [10]. It is also applied in [12] in the form analogous to (13):

$$\Delta^{\alpha} g(t) = \sum_{l=0}^{L} \frac{d_{\alpha_l}}{h^{\alpha_l}} g(t-lh), \tag{19}$$

The coefficients d_{α_l} are computed using the recursive formula (10) and the variable order α:

$$\begin{aligned} d_{\alpha_0} &= 1, \\ d_{\alpha_l} &= \left(1 - \frac{1+\alpha_l}{l}\right) d_{\alpha_{l-1}}, \quad l = 1, \ldots, L. \end{aligned} \tag{20}$$

Next, assume that the quasi time constant T can also take various values in discrete time instants l with the period h:

$$T = \{T_0, T_1, \ldots, T_L\}, \; T_l > 0, \; l = 0, 1, \ldots, L. \tag{21}$$

After applying (18) and (21) to the operator (19) the differential equation (7) turns into the following discrete form:

$$\sum_{l=0}^{L} a_l g(k-l) = -g(k) + u(k), \tag{22}$$

where:

$$a_l = \frac{T_l d_{\alpha_l}}{h^{\alpha_l}}, \; l = 0, \ldots, L. \tag{23}$$

Assume that the initial condition for the function $g(k)$ is the following discrete function, defined L steps back to the initial time moment:

$$g_0 = \begin{bmatrix} x_0 \\ x_{-1} \\ x_{-2} \\ \ldots \\ x_{-L} \end{bmatrix}. \tag{24}$$

In (24) index 0 denotes the initial condition for $k = 0$ and $-L$ is the initial condition for the previous L. In this paper this function is assumed homogenous: $g_0 = 0$.

The solution of the discrete equation (22) is expressed by the following proposition:

Proposition 1.

$$g(k) = -\frac{1}{1+a_0} \sum_{l=1}^{L} a_l g(k-l) + \frac{1}{1+a_0} u(k). \tag{25}$$

Proof. To calculate of the response (25) we start from (22):

$$\sum_{l=0}^{L} a_l g(k-l) = -g(k) + u(k) \iff$$

$$a_0 g(k) + \sum_{l=1}^{L} a_l g(k-l) = -g(k) + u(k) \iff$$

$$(1+a_0) g(k) + \sum_{l=1}^{L} a_l g(k-l) = u(k) \iff$$

$$(1+a_0) g(k) = -\sum_{l=1}^{L} a_l g(k-l) + u(k) \iff$$

$$g(k) = -\frac{1}{1+a_0} \sum_{l=1}^{L} a_l g(k-l) + \frac{1}{1+a_0} u(k).$$

□

The system (22) is the scalar system with L delays in the state. It can be expressed as the L dimensional system without delay, analogously as in [31]. Such an idea for IO systems is presented also by [32], [33]. Initial problem for FO system using this approach is discussed e.g. in [34].

Consequently the state vector has the following form:

$$x(k) = \begin{bmatrix} g(k) \\ g(k-1) \\ \ldots \\ g(k-L) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \ldots \\ x_L(k) \end{bmatrix}, \tag{26}$$

and consequently (25) can be expressed as the following state and output equations:

$$\begin{cases} x(k) = Ax(k-1) + Bu(k-1), \\ y(k) = Cx(k), \end{cases} \tag{27}$$

The state, control and output matrices are defined as beneath:

$$A = \begin{bmatrix} e_1 & e_2 & \ldots & \ldots & \ldots & e_L \\ 1 & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{L \times L} \tag{28}$$

where:

$$e_l = -\frac{a_l}{1+a_0}, \; l = 1, \ldots, L. \tag{29}$$

$$B = \left[\frac{1}{1+a_0}, 0, \ldots, 0\right]^T, \tag{30}$$

$$C = \begin{bmatrix} 1 & 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}_{1 \times L}. \tag{31}$$

Using (25) and (29) the solution of equation (27) imposed by the initial condition (24) and control $u(k)$ is as follows:

$$y(k) = \sum_{l=1}^{L} e_l x(k-l) + \frac{1}{1+a_0} u(k). \tag{32}$$

The shape of the solution (32) is determined by the initial condition given as the discrete function (24), the control signal $u(k)$ and the time variant parameters T and α. This will be illustrated by the numerical analysis presented in the sequel.

5. THE PRACTICAL STABILITY OF THE MODEL

The FO system is practically stable, if its approximation is asymptotically stable. The considered VOVT model with fixed sets of orders α_l and quasi time constants T_l is the FOBD model and its practical stability can be defined analogically as for model with constant parameters.

The practical stability of the model we deal with is determined by the asymptotic stability of the approximated system (27)-(29). For fixed sets α and T it can be examined using the following theorem.

Theorem 3. *The sufficient condition f the practical stability of the model* (27)-(29)
Consider the practical implementation of the VOVT system described by the state equation (27)-(29).

It will be practically stable, if (but not iff!) the following condition holds:

$$\sum_{l=2}^{L} |e_l| < 1 - e_1, \tag{33}$$

where e_l are described by (29).

Proof. The condition (33) is proven using practical implementation (27)-(29) and Gershgorin criterion. Let D_l denote Gershgorin disc centered at s_l and with radius r_l. The shape of the state matrix (28) points that the strongest limitation of its spectrum will be given by discs associated to rows.

In this case the whole spectrum of the system is located in only two discs. The first one, called D_1 is associated to the 1'st row of the matrix A. It is centered in $s_1 = e_1$ and its radius equals to:

$$r_1 = \sum_{l=2}^{L} |e_l|. \tag{34}$$

The remaining discs $D_{2,\ldots L}$ associated to rows $2, \ldots, L$ are the same. They are centered in $(0,0)$ and with radius equal 1 describe the stability area for each discrete system. Denote they by D_I. This implies that the stability of the considered system is determined only by the disc D_1 and consequently the system will be stable if:

$$D_1 \subset D_I \tag{35}$$

Next, the centers of all discs lie at the real axis. This implies that condition (35) will be met if:

$$\begin{gathered} s_1 + r_1 < 1,\ s_1 > 0 \\ \vee \\ s_1 - r_1 > -1,\ s_1 < 0 \end{gathered} \tag{36}$$

In our case $s_1 > 0$ because:

$$a_1 = -\left(\frac{-\alpha_1 T_1}{h^{\alpha_1}}\right) > 0$$

and

$$1 + a_0 = 1 + \frac{T_0}{h^{\alpha_0}} > 0$$

Then only the 1'st inequality in (36) needs to be tested. Using (29) after some transformations we obtain:

$$\begin{aligned} & s_1 + r_1 < 1 \iff \\ & \frac{a_1}{1+a_0} + \sum_{l=2}^{L} |e_l| < 1 \iff \\ & e_1 + \sum_{l=2}^{L} |e_l| < 1 \iff \\ & \sum_{l=2}^{L} |e_l| < 1 - e_1. \end{aligned}$$

and the proof is completed. □

The general sufficient condition (33) does not allow to explicitly estimate a range of sample time assuring the practical stability for given α and T. Such an estimation can be given for constant order α and varying quasi time constant T. It is described as beneath.

Proposition 2. *The range of the sample time h assuring the practical stability of the model* (27)-(29).
Consider the FO system approximated by the model (27)-(29). *Assume that:*

- *The order is constant in the whole range: $\alpha_0 = \alpha_1 = \ldots = \alpha_L = \alpha$,*
- $T_0 < T_1\alpha + \sum_{l=2}^{L} T_l |d_l|$.

The sample time h assuring the practical stability of the system meets the following condition:

$$h > \left(T_1\alpha + \sum_{l=2}^{L} T_l |d_l| + T_0\right)^{\frac{1}{\alpha}}. \tag{37}$$

Proof. The condition (37) follows directly from general stability condition (33). For constant α it can be written as:

$$\begin{aligned} & \frac{T_1\alpha}{h^\alpha + T_0} + \frac{h^\alpha}{h^\alpha + T_0} \sum_{l=2}^{L} T_l \left|\frac{d_l}{h^\alpha}\right| < 1 \iff \\ & \iff T_1\alpha + \sum_{l=2}^{L} T_l\,|d_l| < h^\alpha + T_0 \iff \\ & \iff T_1\alpha + \sum_{l=2}^{L} T_l\,|d_l| < h^\alpha - T_0 \iff \\ & \iff h > \left(T_1\alpha + \sum_{l=2}^{L} T_l |d_l| + T_0\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{38}$$

□

It is characteristic for the FOBD approximation that the sample time h is bounded from low, not from up.

6. THE ACCURACY OF THE MODEL

The accuracy of the model we deal with can be estimated using its steady-state response to the Heaviside signal. It is given below.

Proposition 3. *Consider the FO VOVT discrete model described by the state equation* (27)-(31). *Its steady-state response y_{ss} to the Heaviside signal $1(t)$ is equal:*

$$y_{ss} = \frac{1}{1 + \sum_{l=1}^{L} a_l} \tag{39}$$

where a_l are described by (23).

Proof. The Z transform of the step response of the system (27) is as follows:

$$Y(z) = C(zI - A)^{-1} B \frac{z}{z-1}. \tag{40}$$

The steady-state value of (40) can be computed using the FVT (16):

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(z-1)Y(z) = C(I-A)^{-1}B. \tag{41}$$

where I is the identity matrix.

The matrix $(I-A)$ is equal:

$$(I-A)^{-1} = \begin{bmatrix} 1-e_1 & -e_2 & \dots & \dots & \dots & -e_L \\ -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{L\times L} \tag{42}$$

Its determinant Δ_L can be computed recursively:

$$\begin{aligned} &\Delta_1 = 1-e_1, \\ &\Delta_l = \Delta_{l-1} - e_l,\ l = 2,\dots,L. \end{aligned} \tag{43}$$

what finally gives:

$$\Delta_L = 1 - \sum_{l=1}^{L} e_l. \tag{44}$$

The form of the control and output matrices (30) and (31) points that to compute the expression (41) we need to know only one element of the inverse matrix $(I-A)^{-1}$. This is the 1'st element of the 1'st row. Denote it by a_{11}. It is equal:

$$a_{11} = \frac{1}{\Delta_L} = \frac{1}{1-\sum_{l=1}^{L} e_l}. \tag{45}$$

With respect to (30) and (31) the steady state response (41) takes the following form:

$$y_{ss} = \frac{1}{1-a_0}\frac{1}{1-\sum_{l=1}^{L} e_l}. \tag{46}$$

After use (29) in (46) we obtain (39) and the proof is completed. □

Remark (39) allows us to compute the range of y_{ss} for linearly varying T. It is as follows.

Remark 1. *Consider the FO VOVT discrete model described by the state equation* (27)-(31). *Assume that the quasi time constant T linearly varies between initial and final values T_0 and T_L:*

$$T_l = T_0 + l\cdot\Delta T,\ l = 1,\dots,L, \tag{47}$$

where:

$$\Delta T = \frac{T_L - T_0}{L}, \tag{48}$$

Then the steady-state response y_{ss} to the Heaviside signal $1(t)$ *is equal:*

$$y_{ss} = \frac{1}{1+T_0S_0+\Delta T S_L} \tag{49}$$

where:

$$S_0 = \sum_{l=0}^{L} d_{\alpha+l}, \tag{50}$$

$$S_L = \sum_{l=1}^{L} l d_{\alpha+l}. \tag{51}$$

The condition (49) describes all possible scenarios of linear variability of T. It can be increasing, then $T_0 < T_L$ and $\Delta T > 0$. It can be constant with $T_0 = T_L$ and $\Delta T = 0$ or it can be decreasing, what implies $T_0 > T_L$ and $\Delta T < 0$.

Next, it is important to note that the range of y_{ss} for linearly varying quasi time constant T cannot be estimated using only border, constant values T_0 and T_L. It is estimated also by rate of change, expressed by ΔT.

The condition (49) allows to give the increase ΔT_{err} assuring the predefined value of error $\delta_c = 1 - y_{ss}$. It is described by the following remark.

Remark 2. *Consider the discrete model FO VOVT described by the state equation* (27)-(31). *Assume that:*

- *The quasi time constant T linearly varies from initial value T_0 with the increase equal to ΔT_{err} to final value T_{Lerr},*
- *the steady-state response is: $y_{ss} \neq 1.0$,*
- *the error of steady-state response is equal: $\delta_c = 1 - y_{ss}$.*

The error of the steady-state response y_{ss} to the Heaviside signal $1(t)$ *is equal to δ_c for the following value of ΔT_{err}:*

$$\Delta T_{err} = \frac{\delta_c - y_{ss}T_0S_0}{y_{ss}S_L} \tag{52}$$

and consequently T_{Lerr} is equal:

$$T_{Lerr} = T_0 + L\cdot\Delta T_{err}. \tag{53}$$

Assuring error $\delta_c = 0.0$ we obtain ΔT_{acc} assuring the accurate value of the steady state response of the model. It is given below.

Remark 3. *Consider the discrete model FO VOVT described by the state equation* (27)-(31). *Assume that the quasi time constant T linearly varies from initial value T_0 with the increase equal to ΔT_{acc} to final value T_{Lacc}.*

The steady-state response y_{ss} to the Heaviside signal $1(t)$ *is accurate (i.e. it is equal* 1.0*) for ΔT_{acc} equal:*

$$\Delta T_{acc} = -T_0\frac{S_0}{S_L}, \tag{54}$$

and value of T_L equal:

$$T_{Lacc} = T_0 + L\cdot\Delta T_{acc}, \tag{55}$$

where S_0 and S_L are expressed by (50) *and* (51) *respectively.*

It is important to note that the parameters ΔT_{err}, ΔT_{acc}, T_{Lerr} and T_{Lacc} are the only ones that can be explicitly determined. The remaining parameters: h, α, and L are "implicit" in the coefficients d_l and a_l.

Comparing equations (52) vs (54) it can be seen that:

$$\lim_{\delta_c \to 0} \Delta T_{err} = \Delta T_{acc}. \tag{56}$$

7. THE POSITIVITY

The internal positivity of the considered, variable-parameters, discrete system is described by the following proposition.

Proposition 4. *(The positivity of the VOVT system)*
Consider the VOVT system described by the model (27)-(29)*. It is internally positive for any quasi time constants $T \subset \mathbb{R}_+$ and orders $\alpha = [\alpha_0, ..., \alpha_L] \subset \mathbb{R}_+$ meeting the following condition:*

$$\alpha_l < l-1, \;\; l = 2, ..., L. \tag{57}$$

Proof. With respect to the condition (17) the approximated, discrete system (27)-(29) will be internally positive iff all entries of the state matrix (28) are non negative.

This condition requires to examine only 1'st row of the state matrix (28) containing elements e_1-e_L, because all remaining elements are non negative.

With respect to (29) we obtain for l'th element ($l = 1, 2, ..., L$):

$$e_l > 0 \iff -\frac{a_l}{1+a_0} > 0 \iff -\frac{T_l d_{\alpha_l}}{(1+a_0)h^{\alpha_l}} > 0 \iff \\ \iff d_{\alpha_l} < 0 \tag{58}$$

The dependence (58) shows that each element e_l will be positive iff element d_{α_l} associated to it is negative.

These elements d_{α_l} for $l = 0, 1, 2, ..., L$ are equal to:

$$\begin{aligned} d_{\alpha_0} &= 1, \\ d_{\alpha_1} &= -\alpha_1, \\ d_{\alpha_2} &= \left(\frac{1-\alpha_2}{2}\right) d_{\alpha_1}, \\ &\dots \\ d_{\alpha_l} &= \left(\frac{l-1-\alpha_l}{l}\right) d_{\alpha_{l-1}}, \end{aligned} \tag{59}$$

The set of equations (59) shows that the 2'nd element is negative for each positive α_1. This implies that the l-th factor in brackets (...) must be positive to keep the negative sign of d_{α_l}. Meeting of this condition for l-th element gives directly inequality (57) and the proof is completed. □

From the general condition (57) follows the following one immediately:

Proposition 5. *(The positivity of the VOVT system with orders $0.0 < \alpha_l < 1.0$, $l = 0, ..., L$)*
Consider the VOVT system described by the model (27)-(29)*. It is internally positive for any quasi time constants $T \subset \mathbb{R}_+$ and $\alpha \subset [0; 1]$.*

The numerical verification of the proposed positivty conditions is presented in the next section.

Table 1. Numbering of tests of all considered properties of the model.

Tested property	tests numbers
Practical stability	1-8
Accuracy	9-20
Positivity	21-28

Table 2. Parameters of the model to tests of the practical stability.

No	α	T	h [s]	(33) met?	stable?
1	[0.25,0.85]	[1,5]	5	Yes	Yes
2	[0.85,0.15]	[50,5]	1	Yes	Yes
3	[1.10,1.70]	[90,100]	1	No	Yes
4	[0.50,0.50]	[20,100]	2	No	No

8. NUMERICAL TESTS

Numerical tests were executed to illustrate the proposed results for linearly varying orders α and quasi time constants T.

For all experiments, the memory length was equal $L = 100$. This value is the compromise between accuracy vs numerical complexity of the FOBD approximation.

All the experiments are numbered. The number of each experiment is given in the left column of each table in this section. Indices of all experiments are given in the table 1.

A. Practical Stability

Initially, the sufficient condition (33) was examined for the T and α linearly varying between the initial value $(\ldots_0)$ and the final value $(\ldots_L)$. The range with $\ldots_0 > \ldots_L$ denotes the increasing function, the range with $\ldots_0 < \ldots_L$ describes the decreasing function and $\ldots_0 = \ldots_L$ is the constant one.

All ranges of tested parameters and sample times h, as well as the results of tests are given in Table 2 and illustrated by Figures 1-4 showing the Gershgorin discs D_1, D_I and the numerically computed spectrum of the state matrix (28).

Experiment No 3 in Table 2 shows that condition (33) is sufficient only because it is not met for stable case. Next, the practical stability is stronger determined by the quasi time constant T than by the order α.

Table 3. Ranges of the sample time h assuring the practical stability computed with the use of (37).

No	const α	T	h_{min} [s]
5	0.85	[1,5]	0.0438
6	0.50	[50,5]	0.0000
7	1.70	[90,100]	17.2260
8	0.50	[5,30]	1.2829

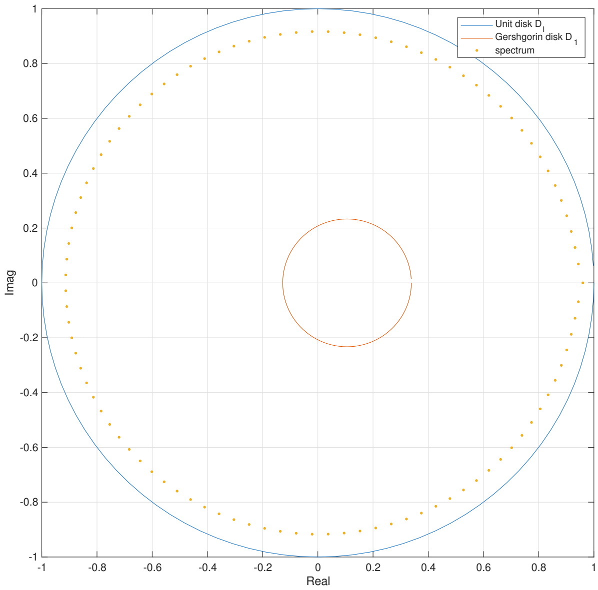

Fig. 1. Stability test No 1

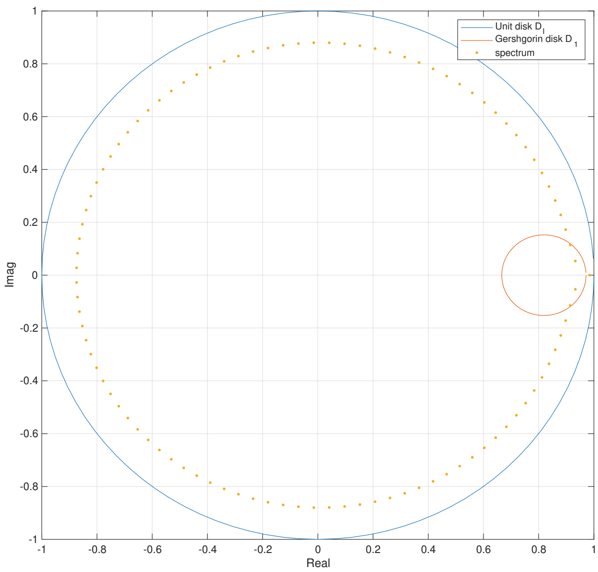

Fig. 2. Stability test No 2

Next ranges of sample time h that ensure practical stability were computed using (37) for constant α and varying quasi time constant T. Results are collected in Table 3.

Results given in the Table 3 are verified by the Figures 5 and 6 that show the spectrum of the system for test No 8 and

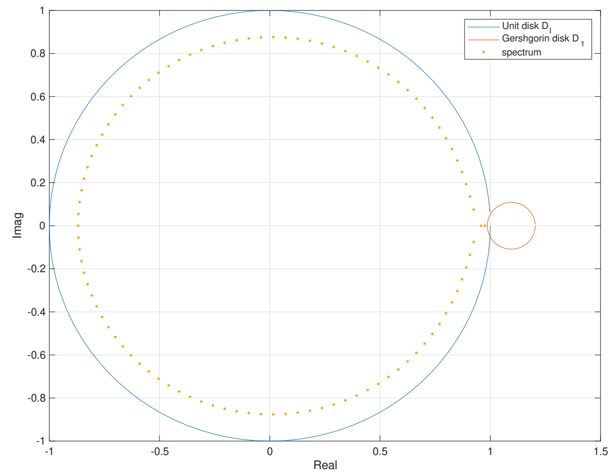

Fig. 3. Stability test No 3

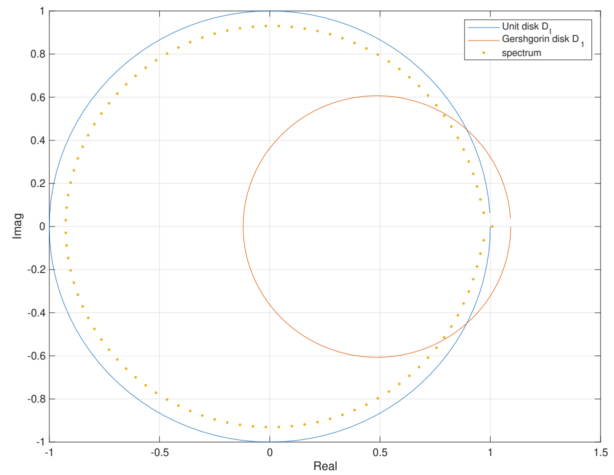

Fig. 4. Stability test No 4

unstable $h = 1.0$ [s] vs stable $h = 2[s]$.

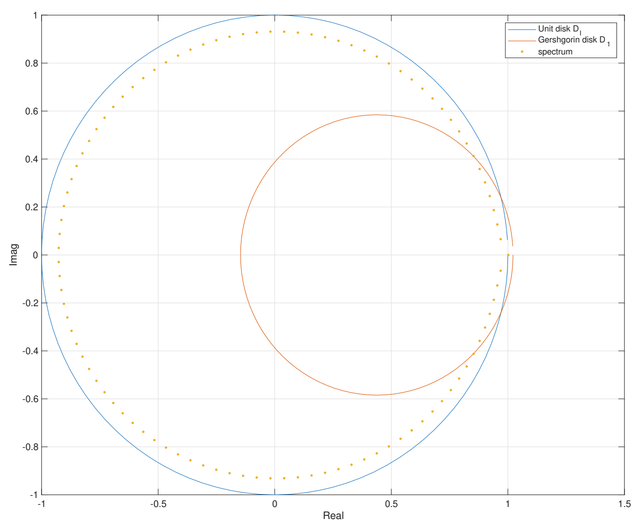

Fig. 5. Stability test No 8, $h = 1.0[s]$, system unstable

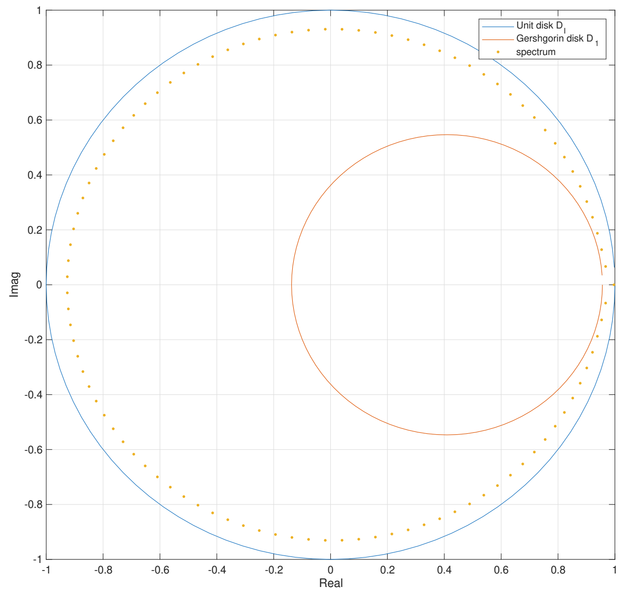

Fig. 6. Stability test No 8, $h = 2.0$ [s], system stable

Table 4. Increments ΔT_{err} assuring the steady-state response equal: $y_{ss} = 0.9$ for various ranges of α and values of T_0.

No	α	T_0	T_{Lerr}	ΔT_{err}
9	[0.25,0.85]	1	3.1342	0.0216
10	[0.50,0.50]	1	0.6127	-0.0039
11	[0.85,0.25]	1	2.1311	0.0316
12	[0.25,0.85]	5	20.3791	0.1553
13	[0.50,0.50]	5	8.6127	0.0365
14	[0.85,0.25]	5	1.8972	-0.0313

B. Accuracy

All tests of accuracy were done using sample time $h = 0.5[s]$ and memory length $L = 100$.

Firstly conditions (52) and (53) describing the steady-state response with assumed accuracy were tested. Results for $\delta_c = 0.1$ are completed in the Table 4 and illustrated by the Figures 7, 8.

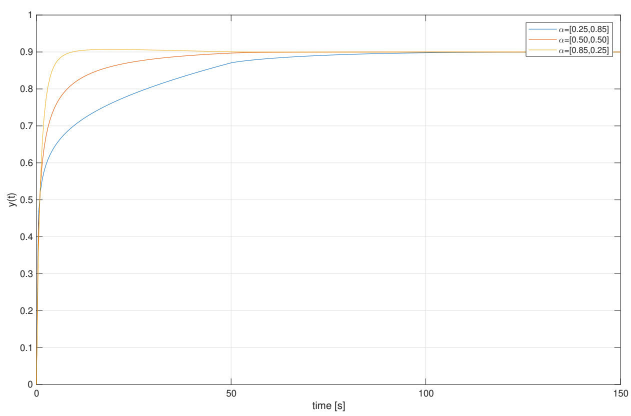

Fig. 7. Accuracy tests No 9-11.

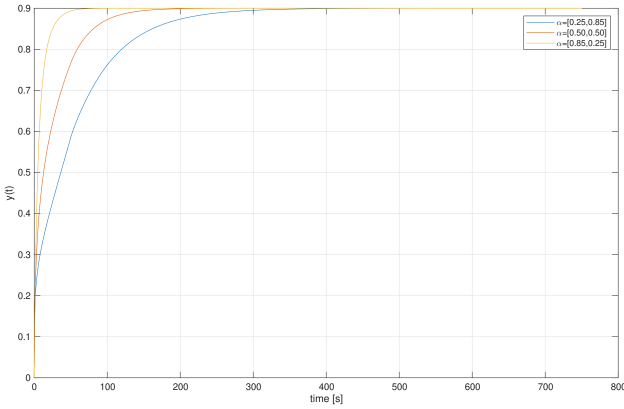

Fig. 8. Accuracy tests No 12-14.

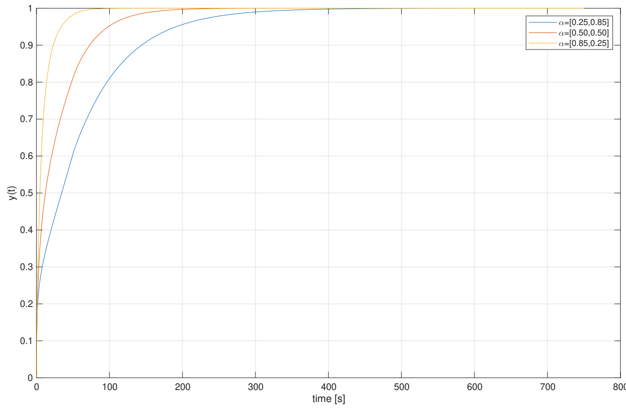

Fig. 10. Accuracy tests No 18-20.

Table 5. Increments ΔT_{acc} assuring the accurate steady-state response for various ranges of α and values of T_0.

No	α	T_0	T_{Lacc}	ΔT_{acc}
15	[0.25,0.85]	1	4.3112	0.0334
16	[0.50,0.50]	1	2.0000	0.0100
17	[0.85,0.25]	1	1.0071	7.1412e-05
18	[0.25,0.85]	5	21.5562	0.1672
19	[0.50,0.50]	5	10.0000	0.0500
20	[0.85,0.25]	5	5.0353	3.5706e-04

Table 6. Ranges of model parameters and the positivity.

No	α	T	Positive?
21	[0.10;0.95]	[1,5]	Yes
22	[0.10;0.95]	[10,20]	Yes
23	[0.95,0.10]	[5,1]	Yes
24	[2.00,1.00]]	[1,10]	No
25	[1.00,2.00]]	[1,10]	No
26	[0.75,1.50]]	[10,20]	Yes
27	[0.75,1.50]]	[10,20]	Yes
28	[1.50,0.75]]	[10,20]	No

The Table 4 shows that the bigger value of T_0 gives the longer range of T.

Next the conditions (54) and (55) describing the accurate steady-state response were examined. They are illustrated by the Table 5 and the Figures 9 and 10.

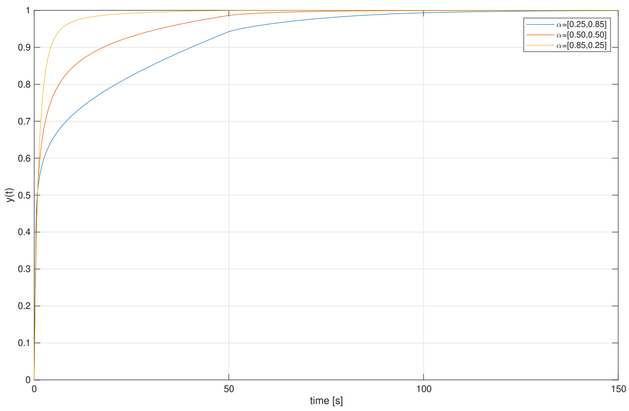

Fig. 9. Accuracy tests No 15-17.

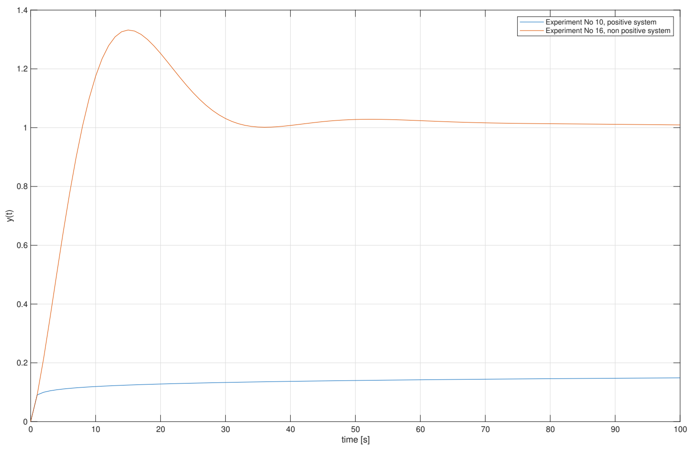

Fig. 11. Step responses for the positive system (test 22) and non positive system (test 28).

C. Positivity

The positivity conditions given in Propositions 3 and 4 will be numerically examined using sample time $h = 1$ [s] and ranges of α and T linearly changing in the predefined range $[\alpha_0; \alpha_L]$ and $[T_0; T_L]$. Results are collected in the Table 6. Step responses for experiments No 22 and 28 are presented in the Figure 11

9. FINAL CONCLUSIONS

The main conclusion of the paper is that the proposed, time variant, scalar state equation is easy to analyse with the use of typical tools delivered by theory of linear, discrete dynamic systems. The time variability of its parameters is described using "long memory effect" modelled by FOBD approximation. The proposed model is the generalization of the variable-parameters transfer function models proposed in [10] and [19].

Next, the proposed approach expands the usefulness of the fractional calculus to various specific cases. For example, it allows to describe variable-structure physical systems and processes.

The area of further investigations of the presented topic is broad. First of all it covers the expanding of the presented results to state equation in general, matrix form.

Next, an effective identification method of parameters α and T should be proposed. It should make possible to identify each particular value α_l or T_l individually, independently on others. This allows to precisely fit a model to real experimental data, but it requires to identify a big amount of parameters, e.g. for memory length L=100 and VOVT model 200 parameters need to be found.

The response of the proposed model to non homogenous initial function g_0, expressed by (24) should be discussed too.

An application of the proposed model to describe of various classes of thermal processes is recently being developed.

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