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On the question of loss characteristics estimation in queueing systems with limited customers' total volume

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Abstract. In this paper, we consider the problem of loss characteristics estimation for queueing systems with random volume customers and limited memory buffer. This issue is of key importance in the processes of analysis and design of many real IT systems with stochastic operation. Main results obtained in this work are formulae allowing to calculate exact values of loss probability and probability of losing a unit of information, or their estimations in the case of systems for which getting exact characteristics is not possible. We focus on two classes of models: first, in which customer's service time and his volume are independent, and second, in which these random variables are dependent. Finally, we present practical results for some special cases of models belonging to these two classes illustrating them with some numerical computations.

Keywords: queueing systems with random volume customers; total customers' volume; loss probability; probability of losing a unit of information; Stieltjes convolution.

1. INTRODUCTION

Queueing systems with random volume customers are a certain generalization of classical models of queuing theory intended primarily to determine the required buffer capacity of communication network nodes (see e.g., [1–3]). The basic assumption of the theory of indicated systems is the heterogeneity of customers due to their different memory volumes that must be stored during their presence in the system. There are two classes of models of systems with random volume customers:

1) models in which service time does not depend on customer volume,
2) models with service time depending on customer volume.

Below, we show that, for first-class models in the case of Poisson entrance flow, knowing the steady-state distribution of customers' number in the classical queueing system is sufficient to determine an analogous distribution in a system with limited buffer capacity when customer volume distribution is known. Moreover, there exists a simple relationship between the distribution functions of the total volume in analogous systems with unlimited and limited memory. This relationship also takes place for second-class models without queue (e.g., egalitarian processor sharing or $M/G/\infty$-type systems) and service time depending on customer volume. If we assume that the above relationship, with some approximation, is also valid for all systems belonging to the second class, then it can be used for approximate estimates of the loss characteristics in these systems.

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Obtaining the values of loss characteristics (e.g., loss probability) was one of the first and most important tasks solved during the analysis of even simple classical queueing models with limited number of servers n and limited number of places in the queue (waiting positions) m (in our notation, which is a modified Kendall's notation, number of places in a queue m does not include customers being on service). In the case, when there are no other limitations (apart from those connected with number of servers and number of places in the queue), steady–state loss probability equals p_{n+m}, where $p_k,\ k=\overline{0,n+m}$ is a steady–state probability of k customers presence in the system. The main result obtained by Erlang in [4] for the $M/M/n/0$-type queueing systems (here $m=0$), generalized later for $M/G/n/0$-type ones (modeling telephone exchanges) was just block probability (which can be understood as the first loss probability formula). For the other classical models with limited n and m, this issue was also very important and was solved both in exact or approximate way (see [5, 6]).

Generalizations of classical queueing models introducing the concept of a customer's volume, started in the 70's of the previous century, also investigate this problem, but this time analyzed task seemed to be much more complicated. Indeed, considering models with random volume customers with limited (by value V, called sometimes memory buffer volume) total customers' volume, we note that customer's loss probability is not equal simply p_{n+m}, because customer's loss takes place even if there are free servers or free places in the queue but his volume is too big to be accepted by the system (more precisely, the sum of the volumes of all customers present in the system during his arrival together with his volume exceeds V). It caused the need of creating new models assuming the other mechanism of losing. In the first period of such investigations (70's and 80's of 20th century), these new models were just small modifications of classical models (see e.g., [7, 8]), but it quickly turn out that such small changes are not enough (simulation results of real telecommunication or computer systems showed that loss characteristics should be different, often much bigger). The cause was the fact that in the first models there was no assumption about the dependence between customer's volume and his service time or scientists took this fact into account only on the level of marginal distributions. Later on, this dependence was taken into account and there appeared many new models (see works [9, 10]) and also works considering loss characteristics calculations in such situations (see again important monograph investigating this problem [1] and e.g., serious works [11, 12]).

This paper presents the analysis of the problem of obtaining loss characteristics formulae for systems with random volume customers, both for those ones in which service time is independent of his volume and for systems in which we have this dependency. The rest of the paper is organized as follows. In the Section 2, we present exact analysis of queueing systems with random volume customers of the $M/G/n/m$-type, limited memory buffer and customer's service time independent of their volume. For such systems, we obtain general exact formulae for p_k distribution, as well as exact formulae for total customers' volume distribution, which allows to calculate exact loss characteristics for many variants of the systems. In Section 3, we investigate an analogous problem for systems with customer volume dependent on his volume, focusing on systems without a queue (egalitarian processor sharing one and $M/G/\infty$-type system). For those systems, we also obtain general formulae for p_k probabilities and total customer volume distribution and show the possibility of using obtained results to calculate loss characteristics and their estimations. Main formulae obtained in these two sections are our own new results that have important meaning for the whole theory of queueing systems with random volume customers. In Section 4, we present some examples of loss characteristics calculations for some special cases of analyzed models. The last Section 5 contains conclusions and final remarks.

2. DETERMINATION OF LOSS CHARACTERISTICS FOR SYSTEMS WITH SERVICE TIME INDEPENDENT OF CUSTOMER'S VOLUME

First, we consider the classical $M/G/n/m$ queueing system, where n is the number of identical servers ($1 \le n \le \infty$), m is the number of waiting places in the queue ($0 \le m \le \infty$). Let $B(t) = \mathbf{P}\{\xi < t\}$ be the distribution function (DF) of service time ξ. Suppose that there exists a finite first moment β_1 of a random variable ξ. Denote by a the parameter of the input stream, let $\rho = a\beta_1/n$. As $\eta(t)$ we denote the number of customers present in the analyzed system at time t. Then the number of customers under service can be determined as $\nu(t) = \min(\eta(t), n)$.

We assume that customers are served in the order in which they arrive at the system, but if there are k customers under service ($k = \overline{1,n}$) at a certain moment of time, they are numbered as $1, 2, \ldots, k$ randomly, which means that any of the possible $k!$ numbering methods can be chosen with the same probability $1/k!$. Obviously, the choice of numbering method of customers under service has no influence on their number in the system.

Denote by $\xi_j^*(t)$ the residual service time of jth customer being on service at time t, $j = \overline{1,\nu(t)}$. Then the behavior of the system under consideration is described by the Markov process

$$\left(\eta(t);\ \xi_l^*(t),\ l = \overline{1,\nu(t)}\right). \tag{1}$$

Let us introduce the following notation for vectors:

$$Y_k = (y_1, \ldots, y_k),\ Y_k^j = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_k),$$

$$(Y_k, z) = (y_1, \ldots, y_k, z).$$

Suppose that each customer, independently of others, is characterized by some random volume ζ, and the distribution function (DF) $L(x) = \mathbf{P}\{\zeta < x\}$ of random variable ζ is determined. We assume that service time ξ of the customer does not depend on his volume. Denote by $\sigma(t)$ the total volume of customers present in the system at time t. Assume also that the total volume $\sigma(t)$ is limited by constant value $V > 0$ that is named the buffer capacity (or buffer memory volume) of the system.

The limitation of total volume leads to additional losses of customers arriving to the system, e.g., let at time τ when there are free servers or free waiting places, a customer of volume x arrives to the system. This customer will be accepted, if $\sigma(\tau^-) + x \le V$. In the other case, he will be lost. The described queueing system with random volume customers and limited buffer capacity we shall denote by $M/G/n/(m,V)$. The behavior of this system is described by the Markov process

$$\left(\eta(t);\ \zeta_i(t),\ i = \overline{1,\eta(t)};\ \xi_l^*(t),\ l = \overline{1,\nu(t)}\right), \tag{2}$$

where $\zeta_i(t)$ is the volume of ith customer in the system at time t. It is obvious that $\sigma(t) = \sum_{k=1}^{\eta(t)} \zeta_i(t)$. The components $\zeta_i(t)$ and $\xi_l^*(t)$ are absent in (2) when $\eta(t) = 0$.

Process (1), which describes the classical system $M/G/n/m$ (for this system we have $V = \infty$), can be characterized by the functions having the following probabilistic sense:

$$W_k^\infty(Y_l, t) = \mathbf{P}\{\eta(t) = k,\ \xi_j^*(t) < y_j,\ j = \overline{1,l}\},$$
$$l = \min(k, n), \quad k = \overline{1, n+m}.$$

Obviously, for probabilities $P_k^\infty(t) = \mathbf{P}\{\eta(t) = k\}$, $k = \overline{1, n+m}$, we obtain $P_k^\infty(t) = W_k^\infty(\infty_l, t)$, where $\infty_l = \underbrace{(\infty, \ldots, \infty)}_{l \text{ components}}$.
Let $P_0^\infty(t) = \mathbf{P}\{\eta(t) = 0\}$.

Process (2) describes the system $M/G/n/(m,V)$. It can be characterized by the following functions:

$$G_k^V(x, Y_l, t) = \mathbf{P}\{\eta(t) = k,\ \sigma(t) < x,\ \xi_j^*(t) < y_j,\ j = \overline{1,l}\},$$

where $l = \min(k, n)$, $k = \overline{1, n+m}$. We can introduce the functions $W_k^V(Y_l, t)$ for this process with the same probabilistic sense. It is clear that

$$W_k^V(Y_l,t) = G_k^V(V,Y_l,t).$$

Note that the functions $G_k^V(x,Y_l,t)$, $W_k^\infty(Y_l,t)$ and $W_k^V(Y_l,t)$ are symmetrical with respect to permutations of the vector Y_l components thanks to the introduced random numbering of the customers under service.

Suppose that the steady state exists for both systems under consideration, i.e., the following limits exist in the sense of a weak convergence: $\eta(t) \Rightarrow \eta, \sigma(t) \Rightarrow \sigma, \xi_l^*(t) \Rightarrow \xi_i^*, \zeta_i(t) \Rightarrow \zeta_i$.

Process (1) describing the classical system $M/G/n/m$, in steady state, can be characterized by the functions having the following probabilistic sense:

$$w_k^\infty(Y_l) = \mathbf{P}\{\eta = k,\ \xi_j^* < y_j,\ j = \overline{1,l}\} = \lim_{t\to\infty} W_k^\infty(Y_l,t),$$
$$l = \min(k,n), \quad k = \overline{1,n+m}.$$

Obviously, for the steady-state probabilities $p_k^\infty = \mathbf{P}\{\eta = k\} = \lim_{t\to\infty} P_k^\infty(t)$, $k = \overline{1,n+m}$, we obtain $p_k^\infty = w_k^\infty(\infty_l)$, where $\infty_l = \underbrace{(\infty,\ldots,\infty)}_{l \text{ components}}$, let $p_0^\infty = \mathbf{P}\{\eta = 0\} = \lim_{t\to\infty} P_0(t)$.

Process (2), in steady state, can be characterized by the following functions:

$$g_k^V(x,Y_l) = \mathbf{P}\{\eta = k,\ \sigma < x,\ \xi_j^* < y_j,\ j = \overline{1,l}\} = \lim_{t\to\infty} G_k^V(x,Y_l,t),$$

where $l = \min(k,n)$, $k = \overline{1,n+m}$. We can introduce the steady-state functions $w_k^V(Y_l)$ for this process with the same probabilistic sense. It is clear that

$$w_k^V(Y_l) = \lim_{t\to\infty} W_k^V(Y_l,t) = g_k^V(V,Y_l).$$

Note that the functions $g_k^V(x,Y_l), w_k^\infty(Y_l)$ and $w_k^V(Y_l)$ are also symmetrical with respect to permutations of the vector Y_l components thanks to the introduced random numbering of the customers under service.

Denote by $L_*^k(x)$ the k-fold Stieltjes convolution of the DF $L(x)$:

$$L_*^0(x) \equiv 1, \quad L_*^k(x) = \int_0^x L_*^{k-1}(x-u)\,\mathrm{d}L(u), \quad k = 1,2,\ldots.$$

Note that, for process (1), the analogous functions $g_k^\infty(x,Y_l)$ can also be defined if we are interested in steady-state DF $D^\infty(x)$ of the total customers' volume in a system with unlimited buffer capacity. In this case, we have

$$\mathbf{P}\{\sigma < x | \eta = k\} = L_*^k(x)$$

and

$$g_k^\infty(x,Y_l) = \mathbf{P}\{\sigma < x,\ \eta = k\} = p_k^\infty L_*^k(x).$$

Consequently (based on total probability theorem),

$$D^\infty(x) = \mathbf{P}\{\sigma < x\} = p_0^\infty + \sum_{k=0}^{n+m} p_k^\infty L_*^k(x).$$

Taking into consideration the random numbering of the served customers, we can use the additional variable method [5] to write out the equations for the functions $W_k^\infty(Y_l,t)$ characterizing the process (1):

$$\frac{\partial P_0^\infty(t)}{\partial t} = -aP_0^\infty(t) + \left.\frac{\partial W_1^\infty(y,t)}{\partial y}\right|_{y=0}; \tag{3}$$

$$\frac{\partial W_1^\infty(y,t)}{\partial t} - \frac{\partial W_1^\infty(y,t)}{\partial y} + \left.\frac{\partial W_1^\infty(y,t)}{\partial y}\right|_{y=0}$$
$$= aP_0^\infty(t)B(y) - aW_1^\infty(y,t) + 2\left.\frac{\partial W_2^\infty(y,u,t)}{\partial u}\right|_{u=0}; \tag{4}$$

$$\frac{\partial W_k^\infty(Y_k,t)}{\partial t} - \sum_{i=1}^k \left[\frac{\partial W_k^\infty(Y_k,t)}{\partial y_i} - \left.\frac{\partial W_k^\infty(Y_k,t)}{\partial y_i}\right|_{y_i=0}\right]$$
$$= \frac{a}{k}\sum_{i=1}^k W_{k-1}^\infty(Y_k^i,t)B(y_i) - aW_k^\infty(Y_k,t)$$
$$+ (k+1)\left.\frac{\partial W_{k+1}^\infty(Y_k,u,t)}{\partial u}\right|_{u=0}, \quad k = \overline{2,n-1}; \tag{5}$$

$$\frac{\partial W_n^\infty(Y_n,t)}{\partial t} - \sum_{i=1}^n \left[\frac{\partial W_n^\infty(Y_n,t)}{\partial y_i} - \left.\frac{\partial W_n^\infty(Y_n,t)}{\partial y_i}\right|_{y_i=0}\right]$$
$$= \frac{a}{n}\sum_{i=1}^n W_{n-1}^\infty(Y_n^i,t)B(y_i) - aW_n^\infty(Y_n,t)$$
$$+ n\left.\frac{\partial W_{n+1}^\infty(Y_{n-1},u,t)}{\partial u}\right|_{u=0} B(y_n); \tag{6}$$

$$\frac{\partial W_k^\infty(Y_n,t)}{\partial t} - \sum_{i=1}^n \left[\frac{\partial W_k^\infty(Y_n,t)}{\partial y_i} - \left.\frac{\partial W_k^\infty(Y_n,t)}{\partial y_i}\right|_{y_i=0}\right]$$
$$= aW_{k-1}^\infty(Y_n,t) - aW_k^\infty(Y_n,t) + n\left.\frac{\partial W_{k+1}^\infty(Y_{n-1},u,t)}{\partial u}\right|_{u=0} B(y_n),$$
$$k = \overline{n+1,n+m-1}; \tag{7}$$

$$\frac{\partial W_{n+m}^\infty(Y_n,t)}{\partial t} - \sum_{i=1}^n \left[\frac{\partial W_{n+m}^\infty(Y_n,t)}{\partial y_i} - \left.\frac{\partial W_{n+m}^\infty(Y_n,t)}{\partial y_i}\right|_{y_i=0}\right]$$
$$= aW_{n+m-1}^\infty(Y_n,t). \tag{8}$$

The steady-state equations follow from (3)–(8) when $t \to \infty$:

$$0 = -ap_0^\infty + \left.\frac{\partial w_1^\infty(y)}{\partial y}\right|_{y=0}; \tag{9}$$

$$-\frac{\partial w_1^\infty(y)}{\partial y} + \left.\frac{\partial w_1^\infty(y)}{\partial y}\right|_{y=0}$$
$$= ap_0^\infty B(y) - aw_1^\infty(y) + 2\left.\frac{\partial w_2^\infty(y,u)}{\partial u}\right|_{u=0}; \tag{10}$$

$$-\sum_{i=1}^{k}\left[\frac{\partial w_k^{\infty}(Y_k)}{\partial y_i}-\frac{\partial w_k^{\infty}(Y_k)}{\partial y_i}\bigg|_{y_i=0}\right]$$
$$=\frac{a}{k}\sum_{i=1}^{k}w_{k-1}^{\infty}(Y_k^i)B(y_i)-aw_k^{\infty}(Y_k)$$
$$+(k+1)\frac{\partial w_{k+1}^{\infty}(Y_k,u)}{\partial u}\bigg|_{u=0},\quad k=\overline{2,n-1}; \tag{11}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_n^{\infty}(Y_n)}{\partial y_i}-\frac{\partial w_n^{\infty}(Y_n)}{\partial y_i}\bigg|_{y_i=0}\right]$$
$$=\frac{a}{n}\sum_{i=1}^{n}w_{n-1}^{\infty}(Y_n^i)B(y_i)-aw_n^{\infty}(Y_n)$$
$$+n\frac{\partial w_{n+1}^{\infty}(Y_{n-1},u)}{\partial u}\bigg|_{u=0}B(y_n); \tag{12}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_k^{\infty}(Y_n)}{\partial y_i}-\frac{\partial w_k^{\infty}(Y_n)}{\partial y_i}\bigg|_{y_i=0}\right]$$
$$=aw_{k-1}^{\infty}(Y_n)-aw_k^{\infty}(Y_n)+n\frac{\partial w_{k+1}^{\infty}(Y_{n-1},u)}{\partial u}\bigg|_{u=0}B(y_n),$$
$$k=\overline{n+1,n+m-1}; \tag{13}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_{n+m}^{\infty}(Y_n)}{\partial y_i}-\frac{\partial w_{n+m}^{\infty}(Y_n)}{\partial y_i}\bigg|_{y_i=0}\right]=aw_{n+m-1}^{\infty}(Y_n). \tag{14}$$

In steady state, the following boundary conditions take place:

$$aw_k^{\infty}(Y_k)=(k+1)\frac{\partial w_{k+1}^{\infty}(Y_k,u)}{\partial u}\bigg|_{u=0},\ k=\overline{1,n-1}; \tag{15}$$

$$aw_k^{\infty}(Y_n)=n\frac{\partial w_{k+1}^{\infty}(Y_{n-1},u)}{\partial u}\bigg|_{u=0}B(y_n),\ k=\overline{n,n+m-1}. \tag{16}$$

Analogously, we can write out the steady-state equations for the process (2) describing the $M/G/n/(m,V)$ queueing system:

$$0=-ap_0^VL(V)+\frac{\partial w_1^V(y)}{\partial y}\bigg|_{y=0}; \tag{17}$$

$$-\frac{\partial w_1^V(y)}{\partial y}+\frac{\partial w_1^V(y)}{\partial y}\bigg|_{y=0}=ap_0^VB(y)L(V)$$
$$-a\int_0^V g_1^V(V-x,y)\,\mathrm{d}L(x)+2\frac{\partial w_2^V(y,u)}{\partial u}\bigg|_{u=0}; \tag{18}$$

$$-\sum_{i=1}^{k}\left[\frac{\partial w_k^V(Y_k)}{\partial y_i}-\frac{\partial w_k^V(Y_k)}{\partial y_i}\bigg|_{y_i=0}\right]$$
$$=\frac{a}{k}\sum_{i=1}^{k}B(y_i)\int_0^V g_{k-1}^V(V-x,Y_k^i)\,\mathrm{d}L(x)-a\int_0^V g_k^V(V-x,Y_k)\,\mathrm{d}L(x)$$
$$+(k+1)\frac{\partial w_{k+1}^V(Y_k,u)}{\partial u}\bigg|_{u=0},\quad k=\overline{2,n-1}; \tag{19}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_n^V(Y_n)}{\partial y_i}-\frac{\partial w_n^V(Y_n)}{\partial y_i}\bigg|_{y_i=0}\right]$$
$$=\frac{a}{n}\sum_{i=1}^{n}B(y_i)\int_0^V g_{n-1}^V(V-x,Y_n^i)\,\mathrm{d}L(x)$$
$$-a\int_0^V g_n^V(V-x,Y_n)\,\mathrm{d}L(x)$$
$$+n\frac{\partial w_{n+1}^V(Y_{n-1},u)}{\partial u}\bigg|_{u=0}B(y_n); \tag{20}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_k^V(Y_n)}{\partial y_i}-\frac{\partial w_k^V(Y_n)}{\partial y_i}\bigg|_{y_i=0}\right]$$
$$=a\int_0^V g_{k-1}^V(V-x,Y_n)\,\mathrm{d}L(x)-a\int_0^V g_k^V(V-x,Y_n)\,\mathrm{d}L(x)$$
$$+n\frac{\partial w_{k+1}^V(Y_{n-1},u)}{\partial u}\bigg|_{u=0}B(y_n),\quad k=\overline{n+1,n+m-1}; \tag{21}$$

$$-\sum_{i=1}^{n}\left[\frac{\partial w_{n+m}^V(Y_n)}{\partial y_i}-\frac{\partial w_{n+m}^V(Y_n)}{\partial y_i}\bigg|_{y_i=0}\right]$$
$$=a\int_0^V g_{n+m-1}^V(V-x,Y_n)\,\mathrm{d}L(x). \tag{22}$$

In this case, boundary conditions have the following form:

$$a\int_0^V g_k^V(V-x,Y_k)\,\mathrm{d}L(x)=(k+1)\frac{\partial w_{k+1}^V(Y_k,u)}{\partial u}\bigg|_{u=0},$$
$$k=\overline{1,n-1}; \tag{23}$$

$$a\int_0^V g_k^V(V-x,Y_n)\,\mathrm{d}L(x)=n\frac{\partial w_{k+1}^V(Y_{n-1},u)}{\partial u}\bigg|_{u=0}B(y_n),$$
$$k=\overline{n,n+m-1}. \tag{24}$$

Let C be some constant value. The following theorem takes place.

Theorem 1. Let the number p_0^{∞} and functions $w_k^{\infty}(Y_l)$, $k=\overline{1,n+m}$, satisfy equations (9)–(16) and the normalization condition $p_0^{\infty}+\sum_{k=1}^{n}w_k^{\infty}(\infty_k)+\sum_{k=n+1}^{n+m}w_k^{\infty}(\infty_n)=1$. Then the number $p_0^V=Cp_0^{\infty}$ and functions $g_k^V(x,Y_l)=Cw_k^{\infty}(Y_l)L_*^k(x)$, $k=\overline{1,n+m}$, satisfy equations (17)–(24).

The theorem can be proved by direct substitution p_0^V and $g_k^V(x,Y_l)$ in the above-mentioned form into equations (17)–(24). As a result, we obtain equations (9)–(16) for p_0 and functions $w_k^{\infty}(Y_l)$. □

Corollary 1. If p_k^V is the steady-state probability of presence of k customers in the system $M/G/n/m$, then the number $p_k^V = Cp_k^\infty L_*^k(V)$, $k = \overline{0,n+m}$ is the steady-state probability of k customers' presence in the system $M/G/n/(m,V)$, and $g_k^V(x,\infty_l) = Cp_k^\infty L_*^k(x)$, where $x \le V$.

Constant C can be determined from the normalization condition:

$$C = \left[\sum_{k=0}^{n+m} p_k^V L_*^k(V)\right]^{-1}.$$

From the corollary, we have for DF $D^V(x)$ of the total customers' volume of the system with limited buffer capacity:

$$\begin{aligned} D^V(x) = \mathbf{P}\{\sigma < x\} &= p_0^V + \sum_{k=1}^{n} g_k^V(x,\infty_k) + \sum_{k=n+1}^{n+m} g_k^V(x,\infty_n) \\ &= Cp_0^\infty + C\sum_{k=0}^{n+m} p_k^\infty L_*^k(x). \end{aligned}$$

As $D^V(V) = 1 = Cp_0^\infty + C\sum_{k=0}^{n+m} p_k^\infty L_*^k(V)$, we obtain:

$$C = \left[p_0^\infty + \sum_{k=0}^{n+m} p_k^\infty L_*^k(V)\right]^{-1} = [D^\infty(V)]^{-1}.$$

Finally, we have:

$$D^V(x) = \begin{cases} \dfrac{D^\infty(x)}{D^\infty(V)}, & \text{if } x \le V, \\ 1, & \text{if } x > V. \end{cases} \tag{25}$$

It is clear that analogous results hold for the systems $M/G/\infty$ and $M/G/(\infty,V)$ with random volume customers. The probability P_{loss} of a customer loss in the $M/G/n/(m,V)$ system can now be determined from the following equilibrium equation (we build it on the obvious assumption that, in steady state, mean value of number of customers that are accepted by the system (are not lost) during some unit of time equals to mean value of the number of customers served within this unit):

$$a(1-P_{\text{loss}}) = \sum_{i=1}^{n} i \frac{\partial w_i^V(\infty_{i-1},u)}{\partial u}\bigg|_{u=0} + n\sum_{i=n+1}^{n+m} \frac{\partial w_i^V(\infty_{n-1},u)}{\partial u}\bigg|_{u=0},$$

where we obtain:

$$\begin{aligned} P_{\text{loss}} = 1 - \frac{1}{a}\Bigg[&\sum_{i=1}^{n} i \frac{\partial w_i^V(\infty_{i-1},u)}{\partial u}\bigg|_{u=0} \\ &+ n\sum_{i=n+1}^{n+m} \frac{\partial w_i^V(\infty_{n-1},u)}{\partial u}\bigg|_{u=0}\Bigg]. \end{aligned}$$

It is clear that the proved theorem also holds for the case of an unlimited queue ($m=\infty$), if $\rho = a\beta_1/n < 1$. In this case, we can also determine P_{loss} from the following relation (based on total probability theorem):

$$P_{\text{loss}} = 1 - \int_0^V D^V(V-x)\,\mathrm{d}L(x).$$

Accurate calculations in this case are possible if e.g. the customers' volume is exponentially distributed with the parameter f: $L(x) = 1 - e^{-fx}$, $x > 0$. Then we have:

$$L_*^k(x) = 1 - e^{-fV}\sum_{j=0}^{k-1} \frac{(fx)^j}{j!}.$$

A generalization of the exponential distribution is the gamma distribution with parameters $\alpha > 0$, $f > 0$, whose DF is given by the formula:

$$L(x) = L(\alpha,f,x) = \frac{1}{\Gamma(a)}\int_0^x f^\alpha u^{\alpha-1} e^{-fu}\,\mathrm{d}u, \quad x > 0.$$

The Stieltjes convolution of such DFs is also a DF of the gamma function:

$$L_*^k(x) = L(k\alpha,f,x).$$

The other possibility is a uniform distribution on the interval $[a,b]$, where $a \ge 0, b > a$. In this case we get [11]

$$\begin{aligned} L_*^k(x) = &\left(\frac{-1}{b-a}\right)^k \\ &\times \sum_{j=0}^{k} \frac{(-1)^j[(b-a)j - bk + x]^k H((b-a)j - bk + x)}{j!(k-j)!}, \end{aligned}$$

where $H(x)$ is a left-side continuous version of Heaviside unit-step function i.e., $H(x) = 0$ for $x \le 0$ and $H(x) = 1$, otherwise. More interesting is the case of the systems $M/D/1/(\infty,V)$ and $M/D/1/\infty$ with constant service time $\xi \equiv t_0$. Then $\rho = at_0$ and $p_k^V = Cp_k^\infty L_*^k(V)$, $k = 0,1,\ldots$, where

$$\begin{aligned} &p_0^\infty = 1-\rho; \quad p_1^\infty = (1-\rho)(e^\rho - 1); \\ &p_k^\infty = (1-\rho)\left\{e^{k\rho} + \sum_{i=1}^{k-1}(-1)^{k-i}e^{i\rho}\left[\frac{(i\rho)^{k-i}}{(k-i)!} + \frac{(i\rho)^{k-i-1}}{(k-i-1)!}\right]\right\}, \\ &\qquad k \ge 2. \end{aligned}$$

Note that, in this case, steady state exists for the classical system $M/D/1/\infty$ when $\rho < 1$ and the following formula holds: $C\sum_{k=0}^{\infty} p_k^\infty L_*^k(V) = 1$, whence

$$C = \left[\sum_{k=0}^{\infty} p_k^\infty L_*^k(V)\right]^{-1}.$$

The loss probability P_{loss} can be calculated from the formula $\rho(1-P_{\text{loss}}) = 1 - p_0$ and we finally get:

$$P_{\text{loss}} = 1 - \rho^{-1}(1-p_0).$$

Another characteristic of losses is the probability of losing a unit volume (of information) Q_{loss} [1]. It characterizes the relative part of total volume of customers that were lost during infinite time interval. If there are no other limitation in the system except for the total volume, probability Q_{loss} can be calculated as:

$$Q_{\text{loss}} = 1 - \frac{1}{\varphi_1} \int_0^V x D^V (V-x)\,\mathrm{d}L(x),$$

where $\varphi_1 = \mathbf{E}\zeta$ is the first moment of customer's volume. Then, from (25), for the system $M/G/n/(\infty,V)$, we obtain the following relations:

$$P_{\text{loss}} = 1 - \frac{1}{D^\infty(V)} \int_0^V D^\infty(V-x)\,\mathrm{d}L(x); \tag{26}$$

$$Q_{\text{loss}} = 1 - \frac{1}{\varphi_1 D^\infty(V)} \int_0^V x D^\infty(V-x)\,\mathrm{d}L(x). \tag{27}$$

Example 1. Consider system $M/M/1/\infty$ with exponential distributed customer's volume (with parameter f). Then we have (see again [1]):

$$D^\infty(x) = 1 - \rho e^{-(1-\rho)fx},$$

where $\rho = a/u < 1$, a and μ are the parameters of entrance flow and service time distribution, respectively. Then, for the system $M/M/1/(\infty,V)$ with the same parameters, we easily obtain (note that in this case ρ value does not have to be less than 1, limited value of V causes the existence of steady state automatically):

$$P_{\text{loss}} = \begin{cases} \dfrac{1-\rho}{e^{(1-\rho)fV} - \rho}, & \text{if } \rho \neq 1, \\ (1+fV)^{-1}, & \text{if } \rho = 1; \end{cases} \tag{28a}$$

$$Q_{\text{loss}} = \frac{p_0 e^{-fV}}{\rho}\left[(1+\rho)e^{\rho f V} - 1\right], \tag{28b}$$

where

$$p_0 = \begin{cases} \dfrac{1-\rho}{1-\rho e^{-(1-\rho)fV}}, & \text{if } \rho \neq 1, \\ (1+fV)^{-1}, & \text{if } \rho = 1. \end{cases}$$

3. DETERMINATION OF LOSS CHARACTERISTICS FOR SYSTEMS WITHOUT QUEUE AND SERVICE TIME DEPENDENT ON CUSTOMERS'S VOLUME

Let a be the parameter of entrance flow and $\mathbf{E}\xi = \beta_1$ be the mean customer's length (for systems in which only one customer can be served by the same device, it means customers's mean service time; for the other systems, e.g. processor sharing systems $M/G/1/\infty - EPS$, when one device can serve many customers at the same time, it means customer's mean service time under condition that there are no other customers served by the same device [13]. In this section, we first determine the steady–state ($\rho = a\beta_1 < 1, \beta_1 = \mathbf{E}\xi = \int_0^\infty [1-B(t)]\,\mathrm{d}t$, where $B(t)$ is the customer's length distribution function) distribution function (DF) $D^\infty(x) = \mathbf{P}\{\sigma < x\}$ of total customers' volume σ for the system $M/G/1/\infty - EPS$ (egalitarian processor sharing system) with unlimited buffer capacity ($V = \infty$) to compare obtained results with ones for the system $M/G/1/\infty - EPS$ but with limited V.

Let $F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}$ be the joint distribution function of the customer's volume and its length. Then, we obviously have: $L(x) = \mathbf{P}\{\zeta < x\} = F(x,\infty)$, $B(t) = \mathbf{P}\{\xi < t\} = F(\infty,t)$. The behavior of the system can be described by the following Markov process:

$$\left(\eta(t), \xi_j^*(t),\ j = \overline{1,\eta(t)}\right), \tag{29}$$

where $\eta(t)$ is the number of customers present in the system at time t, ξ_j^* is the residual length of jth customer in the system after time t. In steady state we have $\eta(t) \Rightarrow \eta$, $\xi_j^*(t) \Rightarrow \xi_j^*$, where η and ξ_j^* are random variables independent of time variable t whereas the random variable $\sigma = \sum_{j=1}^{\eta} \sigma_j$, where σ_j is the volume of jth customer, $j = \overline{1,\eta}$, is a steady–state total customers' volume.

Further, we shall assume that the customers that are currently in the system are numbered randomly, that is, if at instant time t there are k customers in the system, then any of possible $k!$ numerations can be used with probability $1/k!$.

In steady state, the process (29) is characterized by the following functions:

$$p_k^\infty = \mathbf{P}\{\eta = k\},\ k = 0,1,\ldots; \tag{30}$$

$$\theta_k^\infty(Y_k) = \mathbf{P}\{\eta = k, \xi_j^* < y_j, j = \overline{1,k}\},\ k = 1,2,\ldots. \tag{31}$$

It is clear that $p_k^\infty = \theta_k^\infty(\infty_k)$, $k = 1,2,\ldots$, where $\infty_k = (\infty,\ldots,\infty)$ is a vector having k infinity components. It is also known that (see [14])

$$\theta_k^\infty(Y_k) = p_0 a^k \prod_{j=1}^{k} \int_0^{y_j} [1-B(t)]\,\mathrm{d}t,\ k = 1,2,\ldots, \tag{32}$$

where $p_0 = 1-\rho$.

Let now $D_k^\infty(x|Y_k) = \mathbf{P}\{\sigma < x | \xi_j^* = y_j,\ j = \overline{1,k}\}$ be the conditional DF of RV σ under condition $\xi_j^* = y_j$, $j = \overline{1,k}$. Then, based on total probability theorem, we have:

$$D^\infty(x) = p_0^\infty + \sum_{k=1}^{\infty} \int_0^\infty \ldots \int_0^\infty D_k^\infty(x|Y_k)\,\mathrm{d}\theta_k^\infty(Y_k), \tag{33}$$

where

$$\mathrm{d}\theta_k^\infty(Y_k) = p_0^\infty a^k \prod_{j=1}^{k} [1-B(y_j)]\,\mathrm{d}y_j.$$

It is clear that functions $\theta_k(Y_k)$ are symmetrical according to all permutations of the components of the vector Y_k due to the random numbering of customers currently present in the system. It is known [14] that

$$D_k^\infty(x|Y_k) = E_{y_1} * \ldots * E_{y_k}(x),$$

where

$$E_{y_j} = \frac{H_{y_j}(x)}{1 - B(y_j)}, \; H_{y_j}(x) = L(x) - F(x, y_j). \tag{34}$$

Then, we have from (32)–(34):

$$D^\infty(x) = p_0^\infty \left[1 + \sum_{k=1}^\infty a^k \int_0^\infty \ldots \int_0^\infty H_{y_1} * \ldots * H_{y_k}(x) \, \mathrm{d}y_1 \ldots \mathrm{d}y_k \right]. \tag{35}$$

Let us introduce the following notation: $R(x) = \int_0^\infty H_y(x) \, \mathrm{d}y$. Then, we can rewrite (35) as follows:

$$D^\infty(x) = p_0^\infty \left[1 + \sum_{k=1}^\infty a^k R_*^k(x) \right], \tag{36}$$

where $R_*^k(x)$ is the kth order Stieltjes convolution of functions $R(x)$. Note that formula (36) is satisfied for all $x > 0$ and the function $H_y(x)$ can be presented in the following form:

$$H_y(x) = L(x) - F(x, y) = L(x)[1 - B(y|\zeta < x)],$$

where $B(y|\zeta < x)$ is conditional DF of customer's length under condition $\zeta < x$. From the last relation, we can understand the probabilistic sense of $R(x)$ as follows:

$$R(x) = L(x) \int_0^\infty [1 - B(y|\zeta < x)] \, \mathrm{d}y = L(x) \mathbf{E}(\xi|\zeta < x).$$

where $\mathbf{E}(\xi|\zeta < x)$ is conditional mean value of customer's length under condition $\zeta < x$.

Consider now $M/G/1/(\infty, V) - EPS$ processor sharing system, where V is the buffer capacity of the system, $V < \infty$. The rest of system parameters are the same as in the previous one analyzed above ($M/G/1/\infty - EPS$).

The behavior of $M/G/1/(\infty, V) - EPS$ system can be described by the following Markov process:

$$\left(\eta(t), \sigma_j(t), \xi_j^*(t), \; j = \overline{1, \eta(t)} \right), \tag{37}$$

where $\eta(t)$ is the number of customers present in the system at time t, $\sigma_j(t)$ is the volume of jth customer in the system at this time, and $\xi_j^*(t)$ is the residual length of jth customer after time t. In steady state we have again $\eta(t) \Rightarrow \eta$, $\xi_j^*(t) \Rightarrow \xi_j^*$, $\sigma_j(t) \Rightarrow \sigma_j$, if $V < \infty$. The random value $\sigma = \sum_{j=1}^\eta \sigma_j$ is then the steady–state total customers' volume.

In steady state, process (37) can be characterized by the following functions:

$$g_k^V(x, Y_k) = \mathbf{P}\{\eta = k, \; \sigma < x, \; \xi_j^* < y_j, \; j = \overline{1, k}\}, \quad k = 1, 2, \ldots; \quad x < V; \tag{38}$$

$$\theta_k^V(Y_k) = \mathbf{P}\{\eta = k, \xi_j^* < y_j, \; j = \overline{1, k}\} = g_k^V(V, Y_k), \quad k = 1, 2, \ldots; \tag{39}$$

and constant

$$p_0^V = \mathbf{P}\{\eta = 0\}. \tag{40}$$

It is clear that the functions $g_k^V(x, Y_k)$ and $\theta_k^V(Y_k)$ are also symmetrical according to all permutations of y_j components of the vector Y_k analogously to previously analyzed system with unlimited buffer capacity.

For the functions (38)–(39) and constant (40) we can write out the following equations, taking into consideration the above-mentioned symmetry (see also [14]):

$$0 = -a p_0^V L(V) + \left. \frac{\partial \theta_1^V(y)}{\partial y} \right|_{y=0}; \tag{41}$$

$$-\frac{\partial \theta_1^V(y)}{\partial y} + \left. \frac{\partial \theta_1^V(y)}{\partial y} \right|_{y=0} = a p_0^V F(V, y) - a \int_0^V g_1^V(V - x, y) \, \mathrm{d}L(x) + \left. \frac{\partial \theta_2^V(y, u)}{\partial u} \right|_{u=0}; \tag{42}$$

$$-\frac{1}{k} \left[\frac{\partial \theta_k^V(Y_k)}{\partial y_j} - \left. \frac{\partial \theta_k^V(Y_k)}{\partial y_j} \right|_{y_j=0} \right] = a \int_0^V g_{k-1}^V(V - x, Y_{k-1}) \, \mathrm{d}_x F(x, y_j) - a \int_0^V g_k^V(V - x, Y_k) \, \mathrm{d}L(x) + \left. \frac{\partial \theta_{k+1}^V(Y_k, u)}{\partial u} \right|_{u=0}, \quad k = 2, 3, \ldots; \tag{43}$$

In steady state, the following equilibrium equations, playing the role of boundary conditions for equations (41)–(43), are satisfied:

$$a \int_0^V g_k^V(V - x, Y_k) \, \mathrm{d}L(x) = \left. \frac{\partial \theta_{k+1}^V(Y_k, u)}{\partial u} \right|_{u=0}, \quad k = 1, 2, \ldots \tag{44}$$

The solution of system of equations (41)–(43) for the case $V < \infty$ can be written down, if we introduce the following functions:

$$\Phi_y^V(x) = \int_0^y H_u(x) \, \mathrm{d}u, \quad \text{where } 0 < x \le V.$$

The probabilistic sense of $\Phi_y^V(x)$ is clear, if we write it as

$$\Phi_y^V(x) = L(x)\int_0^y [1 - B(u|\zeta < x)]\,\mathrm{d}u. \tag{45}$$

Now, we can write down the solution in the form:

$$g_k^V(x, Y_k) = Ca^k \Phi_{y_1}^V * \ldots * \Phi_{y_k}^V(x), \quad k = 1, 2, \ldots; \quad 0 < x \le V. \tag{46}$$

Then, we have from (39):

$$\theta_k^V(Y_k) = Ca^k \Phi_{y_1}^V * \ldots * \Phi_{y_k}^V(V), \quad k = 1, 2, \ldots. \tag{47}$$

Steady-state probabilities $p_k^V = \mathbf{P}\{\eta = k\}$, $k = 1, 2, \ldots$, take the form:

$$p_k^V = \theta_k^V(\infty_k) = Ca^k R_*^k(V), \tag{48}$$

where

$$R(x) = \lim_{y\to\infty} \Phi_y^V(x) = L(x)\int_0^\infty [1 - B(u|\zeta < x)]\,\mathrm{d}u,\ x \le V, \tag{49}$$

If we use (45), (47) and (41), we can note that $p_0^V = C$, so formula (48) is also valid for $k = 0$. Then C constant can be found from the normalization condition:

$$C = p_0^V = \left[\sum_{k=0}^\infty a^k R_*^k(V)\right]^{-1}.$$

Obviously $g_k^V(x, \infty_k) = p_0^V a^k R_*^k(x)$, $x \le V$. Hence, we obtain (based on total probability theorem) for $0 < x \le V$:

$$D^V(x) = \mathbf{P}\{\sigma < x\} = p_0^V + \sum_{k=1}^\infty g_k^V(x, \infty_k) = p_0^V\left[1 + \sum_{k=1}^\infty a^k R_*^k(x)\right].$$

As a result, we get:

$$D^V(x) = \begin{cases} p_0^V\left[1 + \sum_{k=1}^\infty a^k R_*^k(x)\right], & \text{if } x \le V, \\ 1, & \text{if } x > V. \end{cases} \tag{50}$$

Comparing (50) and (36), we come to conclusion that function $D^V(x)$ is a truncation of function $D^\infty(x)$ on the interval $[0, V]$. In other words:

$$D^V(x) = \begin{cases} \dfrac{D^\infty(x)}{D^\infty(V)}, & \text{if } x \le V, \\ 1, & \text{if } x > V. \end{cases} \tag{51}$$

Further, for $V = \infty$, we can easily obtain from (48), taking into account also normalization condition $\sum_{k=0}^\infty p_k^\infty = 1$ and relation $\lim_{V\to\infty} R_*^k(V) = \beta_1^k$ ($\beta_1 = \mathbf{E}\xi$ is the first moment of customer's length), well known result:

$$p_k^\infty = C\rho^k = (1-\rho)\rho^k,$$

where $\rho = a\beta_1$.

It can be easily shown, by analogous way, that formula (51) is also valid for system $M/G/(\infty, V)$.

Hence, we can use relations (26) and (27) for P_{loss} and Q_{loss} calculations for specified systems, if the explicit form of $D^\infty(x)$ is known, and there are no other limitations other than the buffer capacity.

Denote by $\delta(s) = \int_0^\infty e^{-sx}\,\mathrm{d}D^\infty(x)$ Laplace-Stieltjes transform (LST) of the steady–state function $D^\infty(x)$ and by $\alpha(s,q) = \int_0^\infty\int_0^\infty e^{-sx-qt}\,\mathrm{d}F(x,t)$ double LST of the function $F(x,t)$. It is known [14] that:

$$\delta(s) = \frac{1-\rho}{1 + a\alpha_q'(s,q)|_{q=0}}. \tag{52}$$

Example 2. Consider the following special case of $M/G/1/\infty$–EPS queueing system. Let customer's volume ζ have an exponential distribution with parameter $f > 0$: $L(x) = 1 - e^{-fx}$ and its first moment is denoted as φ_1. Assume that customer's length ξ can be determined as $\xi = c\zeta + \xi_1$, $c \ge 0$, where random variables ξ_1 and ζ are independent. Let $\kappa_1 = \mathbf{E}\xi_1$. In this case, we can determine the explicit form of DF $D^\infty(x)$, if $\rho = a\beta_1 = a(c\varphi_1 + \kappa_1) < 1$ (see again [14]):

$$D^\infty(x) = 1 - \frac{(1-\rho)e^{-fx}}{2b}\left[\frac{(b+\rho_2)^2 e^{\frac{(\rho_2+b)fx}{2}}}{2-\rho_2-b} - \frac{(b-\rho_2)^2 e^{\frac{(\rho_2-b)fx}{2}}}{2-\rho_2+b}\right], \tag{53}$$

where $\rho = \rho_1 + \rho_2$, $\rho_1 = ac/f$, $\rho_2 = a\kappa_1$, $b = \sqrt{\rho_2^2 + 4\rho_1}$.

Denote by

$$I = \int_0^V D^\infty(V-x)\,\mathrm{d}L(x) = f\int_0^V D^\infty(V-x)e^{-fx}\,\mathrm{d}x,$$

$$J = \frac{1}{\varphi_1}\int_0^V xD^\infty(V-x)\,\mathrm{d}L(x) = f^2\int_0^V xD^\infty(V-x)e^{-fx}\,\mathrm{d}x.$$

Then, we obtain from (53) that:

$$I = 1 - e^{-fV} + \frac{1-\rho}{b}e^{-fV}\times \left\{\frac{b+\rho_2}{2-\rho_2-b}\left[1 - e^{(b+\rho_2)fV/2}\right] + \frac{b-\rho_2}{2-\rho_2+b}\left[1 - e^{-(b-\rho_2)fV/2}\right]\right\},$$

$$J = 1 - e^{-fV}\left\{1 + fV + \frac{1-\rho}{b}\left[\frac{2e^{\frac{(b+\rho_2)fV}{2}} - 2 - (b+\rho_2)fV}{2-\rho_2-b} - \frac{2e^{\frac{-(b-\rho_2)fV}{2}} - 2 + (b-\rho_2)fV}{2-\rho_2+b}\right]\right\}.$$

Then $P_{\text{loss}} = 1 - I/D^{\infty}(V)$, $Q_{\text{loss}} = 1 - J/D^{\infty}(V)$. In the case of $\xi_1 \equiv 0$ ($\xi = c\zeta$, $c > 0$, $\rho = \rho_1$), we easily obtain from (52) that:

$$D^{\infty}(x) = 1 - \sqrt{\rho}e^{-fx}\left[\sinh(\sqrt{\rho}fx) + \sqrt{\rho}\cosh(\sqrt{\rho}fx)\right],$$
$$I = 1 - e^{-fV}\left[\cosh(\sqrt{\rho}fV) + \sqrt{\rho}\sinh(\sqrt{\rho}fV)\right],$$
$$J = 1 - \frac{e^{-fV}}{\sqrt{\rho}}\left[\sqrt{\rho}\cosh(\sqrt{\rho}fV) + \sinh(\sqrt{\rho}fV)\right],$$

whereas we can obtain corresponding formulae for P_{loss} and Q_{loss}. More precisely:

$$P_{\text{loss}} = 1 - \frac{1 - e^{-fV}\left[\cosh(\sqrt{\rho}fV) + \sqrt{\rho}\sinh(\sqrt{\rho}fV)\right]}{1 - \sqrt{\rho}e^{-fV}\left[\sinh(\sqrt{\rho}fV) + \sqrt{\rho}\cosh(\sqrt{\rho}fV)\right]}, \tag{54a}$$

$$Q_{\text{loss}} = 1 - \frac{1 - \frac{e^{-fV}}{\sqrt{\rho}}\left[\sqrt{\rho}\cosh(\sqrt{\rho}fV) + \sinh(\sqrt{\rho}fV)\right]}{1 - \sqrt{\rho}e^{-fV}\left[\sinh(\sqrt{\rho}fV) + \sqrt{\rho}\cosh(\sqrt{\rho}fV)\right]}. \tag{54b}$$

In the case of $c = 0$ ($\rho = \rho_2 = a\kappa_1$) we obtain:

$$D^{\infty}(x) = 1 - \rho e^{-(1-\rho)fx},$$
$$I = 1 - e^{-(1-\rho)fV},$$
$$J = 1 - \frac{e^{-(1-\rho)fV}}{\rho},$$

whereas we obtain for P_{loss} and Q_{loss} relations (28a) and (28b).

It can be easily shown that formulae (54a) and (54b) are also valid for $\rho > 1$, whereas for $\rho = 1$, we can obtain the following results:

$$P_{\text{loss}} = \frac{2(1 + e^{2fV})}{1 + e^{2fV}(3 + 2fV)}, \tag{54c}$$

$$Q_{\text{loss}} = \frac{4e^{-fV}\left[\cosh(fV) + \sinh(fV)\right]}{3 + e^{-2fV} + 2fV}. \tag{54d}$$

If we cannot invert (52), we can very often approximate DF $D^{\infty}(x)$, knowing $p_0 = 1 - \rho$ and its first two moments $\mathbf{E}\sigma^{\infty} = -\delta'(0)$ and $\mathbf{E}(\sigma^{\infty})^2 = \delta''(0)$:

$$\mathbf{E}\sigma^{\infty} = \frac{a\alpha_{11}}{1-\rho}; \quad \mathbf{E}(\sigma^{\infty})^2 = \frac{a\alpha_{21}}{1-\rho} + 2(\mathbf{E}\sigma^{\infty})^2,$$

where α_{ij}, $i,j = 1,2,\ldots$, is the mixed $(i+j)$th moment of random vector (ζ,ξ). Then we can choose as approximation the following function:

$$D_*^{\infty}(x) = p_0 + (1 - p_0)\frac{\gamma(p, gx)}{\Gamma(p)}, \tag{55}$$

where $\gamma(p, gx) = \int\limits_0^{gx} t^{p-1}e^{-t}\,\mathrm{d}t$ is incomplete gamma function and $\Gamma(p) = \gamma(p,\infty)$ is gamma function, p and g are parameters that we define assuming that the first and second moments δ_1^* and δ_2^* of DF $D_*^{\infty}(x)$ must be equal to $\mathbf{E}\sigma^{\infty}$ and $\mathbf{E}(\sigma^{\infty})^2$, respectively. It is easy to show that $\delta_1^* = \frac{(1-p_0)p}{g}$, $\delta_2^* = \frac{(1-p_0)p(p+1)}{g^2}$. Then, from the equalities $\delta_1^* = \mathbf{E}\sigma^{\infty}$ and $\delta_2^* = \mathbf{E}(\sigma^{\infty})^2$, it follows that (see e.g., [1]):

$$p = \frac{(\mathbf{E}\sigma^{\infty})^2}{(1-p_0)\mathbf{E}(\sigma^{\infty})^2 - (\mathbf{E}\sigma^{\infty})^2}; \quad g = \frac{(1-p_0)\mathbf{E}\sigma^{\infty}}{(1-p_0)\mathbf{E}(\sigma^{\infty})^2 - (\mathbf{E}\sigma^{\infty})^2}. \tag{56}$$

4. NUMERICAL RESULTS

In this section, we shall focus on some exemplary calculations of loss characteristics for queueing systems belonging to two the above-mentioned classes. We show computations both for systems in which customer's volume and his service time are independent (first class), and for systems in which these random variables are dependent (second class). We illustrate obtained results with some tables and graphs, presenting exact of two main loss characteristics P_{loss} and Q_{loss}, when obtaining of exact characteristics is possible, or their estimations in opposite case.

4.1. System $M/M/1/(\infty,V)$ with customer's service time independent of his volume

For this system, we assume that customer's service time ξ and his volume ζ are independent, customers arrive at the system according to Poisson flow (time intervals between whiles of customers' arrival are exponentially distributed with the same parameter a), customer's service time is exponentially distributed (with parameter μ), total customers' volume is limited by value V and we have only one server.

If we additionally assume that customer's volume is exponentially distributed with parameter f, then for such queueing systems we can use relations for P_{loss} and Q_{loss} discussed in Section 2 (formulae (28a) and (28b)).

In Tables 1–4, we present the results of calculations of loss characteristics for five different values of ρ, and two different values of f parameters, and on Figs. 1–10, we present graphs of these characteristics treated as the functions of V variable.

If we analyze formulae (28a), (28b), our calculations and graphs, we can see that:

- If $\rho \leq 1$, both P_{loss} and Q_{loss} characteristics converge to 0 when $V \to \infty$, but if $\rho > 1$, we have different, very interesting, convergence, namely:

$$\lim_{V\to\infty} P_{\text{loss}} = 1 - \frac{1}{\rho}, \quad \lim_{V\to\infty} Q_{\text{loss}} = 1 - \frac{1}{\rho^2};$$

- Parameter ρ has an influence on the above-mentioned convergences, the bigger value of ρ is, the slower this convergence is (in case when $\rho \leq 1$); if $\rho > 1$, we have opposite situation, the bigger value of ρ is, the faster convergence is;
- Parameter f also has influence on loss characteristics, the bigger value of f is, the faster convergence is (bigger value of f means smaller mean value of customer's volume, which causes smaller values of loss characteristics).

Table 1

P_{loss} characteristics for $M/M/1/(\infty,V)$ system, $f=1$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.394956	0.448519	0.487399	0.500000	0.612700
2.0	0.168314	0.246072	0.311136	0.333333	0.536289
3.0	0.073915	0.147053	0.222292	0.250000	0.512765
4.0	0.032878	0.091890	0.168969	0.200000	0.504621
5.0	0.014706	0.058918	0.133561	0.166667	0.501690
6.0	0.006595	0.038376	0.108446	0.142857	0.500620
7.0	0.002960	0.025245	0.089787	0.125000	0.500228
8.0	0.001330	0.016714	0.075441	0.111111	0.500084
9.0	0.000597	0.011112	0.064119	0.100000	0.500031
10.0	0.000268	0.007408	0.054997	0.090909	0.500011
15.0	$4.92\cdot10^{-6}$	0.000993	0.027920	0.062500	0.500000
20.0	$9.00\cdot10^{-8}$	0.000134	0.015411	0.047619	0.500000

Table 2

Q_{loss} characteristics for $M/M/1/(\infty,V)$ system, $f=1$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.752923	0.785796	0.808770	0.816060	0.877590
2.0	0.445762	0.532666	0.599700	0.621550	0.799520
3.0	0.240664	0.351627	0.452680	0.487550	0.768510
4.0	0.123402	0.231146	0.351580	0.396340	0.756850
5.0	0.061187	0.152227	0.280310	0.332210	0.752520
6.0	0.029637	0.100588	0.228400	0.285360	0.750930
7.0	0.014113	0.066689	0.189370	0.249890	0.750340
8.0	0.006636	0.044340	0.159200	0.222180	0.750130
9.0	0.003090	0.029547	0.135340	0.199990	0.750050
10.0	0.001429	0.019723	0.116100	0.181810	0.750020
15.0	$2.83\cdot10^{-5}$	0.002648	0.058940	0.125000	0.750000
20.0	$5.32\cdot10^{-7}$	0.000358	0.032530	0.095240	0.750000

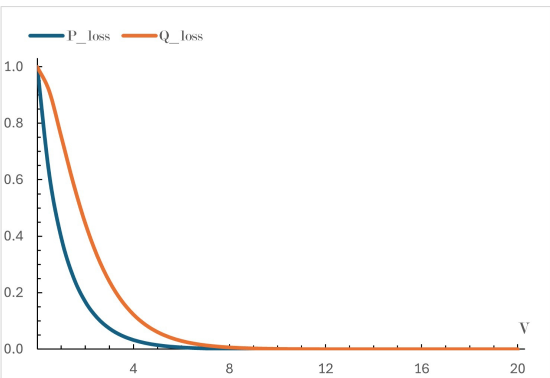

Fig. 1. Loss characteristics for $M/M/1/(\infty,V)$ queueing system, $f=1$, $\rho=0.2$

Table 3

P_{loss} characteristics for $M/M/1/(\infty,V)$ system, $f=0.2$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.821768	0.827665	0.831937	0.833333	0.846547
2.0	0.679620	0.697458	0.710173	0.714286	0.752061
3.0	0.564942	0.595904	0.617907	0.625000	0.689090
4.0	0.471564	0.514716	0.545592	0.555556	0.644882
5.0	0.394956	0.448519	0.487399	0.500000	0.612700
6.0	0.331717	0.393672	0.439567	0.454545	0.588649
7.0	0.279246	0.347623	0.399562	0.416667	0.570320
8.0	0.235527	0.308527	0.365616	0.384615	0.556142
9.0	0.198971	0.275021	0.336454	0.357143	0.545048
10.0	0.168314	0.246072	0.311136	0.333333	0.536289
15.0	0.073915	0.147053	0.222292	0.250000	0.512765
20.0	0.032878	0.091890	0.168969	0.200000	0.504621

Table 4

Q_{loss} characteristics for $M/M/1/(\infty,V)$ system, $f=0.2$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.982878	0.983652	0.984210	0.984391	0.986092
2.0	0.940879	0.945489	0.948731	0.949771	0.959130
3.0	0.884359	0.896164	0.904377	0.906993	0.929861
4.0	0.820183	0.841746	0.856730	0.861484	0.902223
5.0	0.752923	0.785796	0.808773	0.816060	0.877590
6.0	0.685612	0.730424	0.762113	0.772184	0.856273
7.0	0.620229	0.676873	0.717590	0.730585	0.838139
8.0	0.558023	0.625852	0.675607	0.691578	0.822877
9.0	0.499738	0.577730	0.636310	0.655250	0.810125
10.0	0.445762	0.532666	0.599698	0.621555	0.799523
15.0	0.240664	0.351627	0.452684	0.487553	0.768511
20.0	0.122396	0.231146	0.351582	0.396337	0.756847

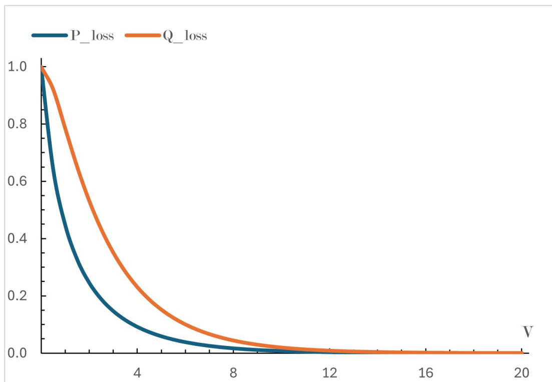

Fig. 2. Loss characteristics for $M/M/1/(\infty,V)$ queueing system, $f=1$, $\rho=0.6$

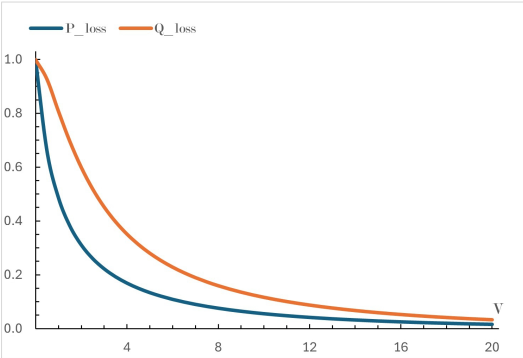

Fig. 3. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 1$, $\rho = 0.9$

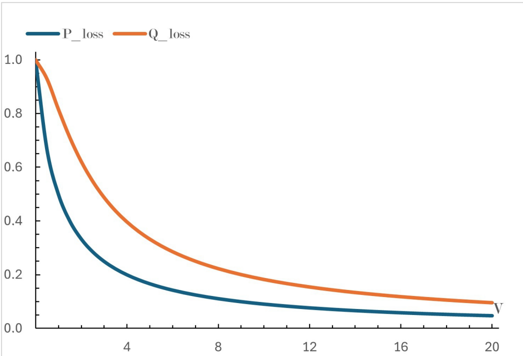

Fig. 4. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 1$, $\rho = 1$

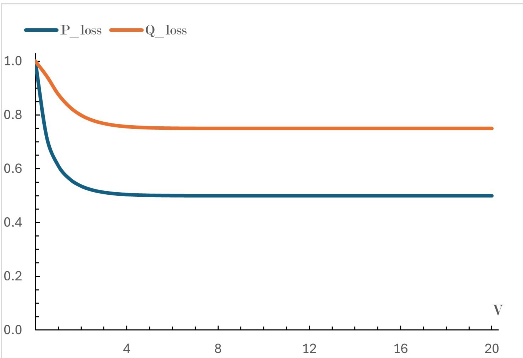

Fig. 5. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 1$, $\rho = 2$

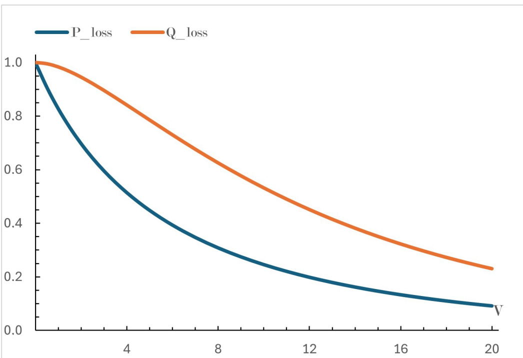

Fig. 6. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 0.2$, $\rho = 0.2$

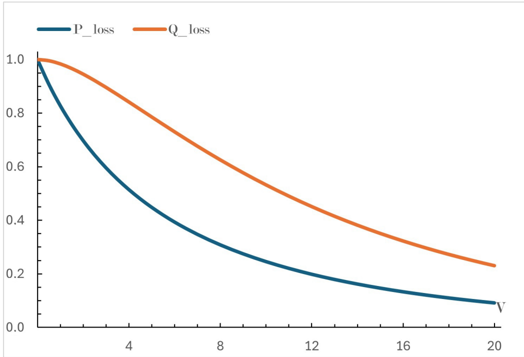

Fig. 7. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 0.2$, $\rho = 0.6$

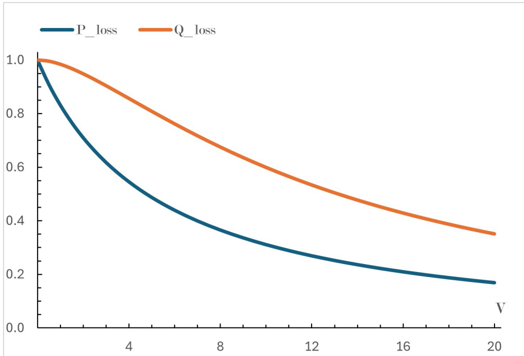

Fig. 8. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 0.2$, $\rho = 0.9$

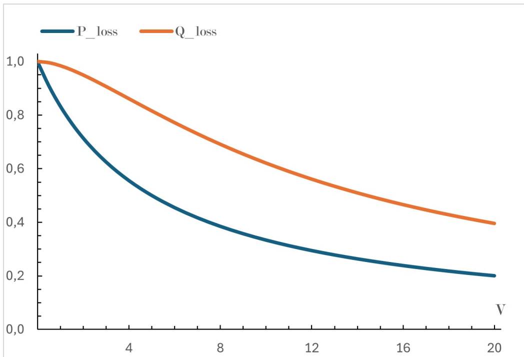

Fig. 9. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 0.2$, $\rho = 1$

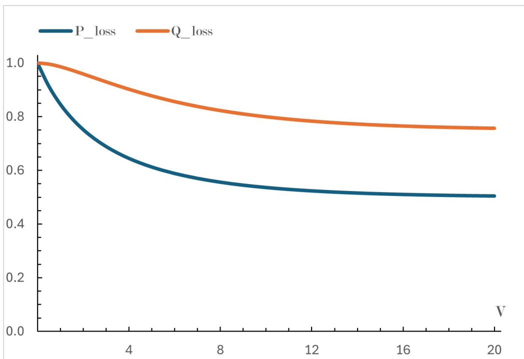

Fig. 10. Loss characteristics for $M/M/1/(\infty, V)$ queueing system, $f = 0.2$, $\rho = 2$

4.2. System $M/M/1/(\infty, V) - EPS$ with customer's length proportional to his volume

Now we shall investigate some special case of processor-sharing system discussed in Section 3. In this case we assume that customer's length ξ and his volume ζ are proportional with coefficient c ($c > 0$), customers also arrive to the system according to Poisson flow with parameter a, total customers' volume is limited by value V and we have only one device servicing all customers at the same time.

Let us additionally assume that customer's volume is exponentially distributed with parameter f. Then for such queueing systems we can use relations for P_{loss} and Q_{loss} using formulae (54a)–(54d).

In Tables 5–8, we present the results of calculations of loss characteristics for five different values of $\rho = \dfrac{ac}{f}$, and two different values of f parameters, and in Figs. 11–20, we present graphs of these characteristics treated as the functions of V variable.

If we analyze formulae (54a)–(54d), calculations presented in tables and graphs, we can conclude that:

- If $\rho \le 1$, loss characteristics converge to 0 when $V \to \infty$, but if $\rho > 1$, we obtain that:
$$\lim_{V\to\infty} P_{\text{loss}} = 1 - \frac{1}{\sqrt{\rho}}, \quad \lim_{V\to\infty} Q_{\text{loss}} = 1 - \frac{1}{\rho};$$
- Parameter ρ has an influence on the above-mentioned convergences, bigger value of ρ causes slower convergence (in case when $\rho \le 1$); if $\rho > 1$ bigger value of ρ causes faster convergence;
- Parameter f also has influence on loss characteristics; a bigger value of f causes faster convergence.

Table 5

P_{loss} characteristics for $M/M/1/(\infty, V) - EPS$ system, $f = 1$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.384662	0.415160	0.435708	0.442166	0.497907
2.0	0.171766	0.235861	0.277412	0.290188	0.394643
3.0	0.086584	0.157754	0.207160	0.222712	0.352546
4.0	0.046693	0.112813	0.164815	0.181874	0.329738
5.0	0.026034	0.083398	0.135818	0.153853	0.316245
6.0	0.014749	0.062909	0.114621	0.133334	0.307919
7.0	0.008420	0.048111	0.098451	0.117647	0.302653
8.0	0.004825	0.037162	0.085722	0.105263	0.299271
9.0	0.002770	0.028917	0.075453	0.095238	0.297077
10.0	0.001592	0.022628	0.067003	0.086957	0.295645
15.0	0.000100	0.006965	0.040490	0.060606	0.293237
20.0	$6.32 \cdot 10^{-6}$	0.002221	0.026788	0.046512	0.292937

Table 6

Q_{loss} characteristics for $M/M/1/(\infty, V) - EPS$ system, $f = 1$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.745578	0.763322	0.775199	0.778917	0.810683
2.0	0.445835	0.514064	0.556966	0.569937	0.671755
3.0	0.255421	0.357546	0.424058	0.444322	0.601731
4.0	0.145425	0.257862	0.338370	0.363625	0.562893
5.0	0.082934	0.190971	0.278962	0.307691	0.539863
6.0	0.047423	0.144110	0.235440	0.266667	0.525650
7.0	0.027177	0.110220	0.202228	0.235294	0.516661
8.0	0.015598	0.085137	0.176081	0.210526	0.510888
9.0	0.008961	0.066249	0.154987	0.190476	0.507143
10.0	0.005151	0.051840	0.137631	0.173913	0.504697
15.0	0.000324	0.015957	0.083169	0.121212	0.500587
20.0	$2.04 \cdot 10^{-5}$	0.005088	0.055026	0.093023	0.500074

Table 7

P_{loss} characteristics for $M/M/1/(\infty, V) - EPS$ system, $f = 0.5$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.610697	0.618714	0.624466	0.626336	0.643848
2.0	0.384662	0.415160	0.435708	0.442166	0.497907
3.0	0.252265	0.303482	0.336779	0.347049	0.431985
4.0	0.171766	0.235861	0.277412	0.290188	0.394643
5.0	0.120564	0.190564	0.237059	0.251473	0.370049
6.0	0.086584	0.157754	0.207160	0.222712	0.352546
7.0	0.063215	0.132676	0.183767	0.200164	0.339577
8.0	0.046693	0.112813	0.164815	0.181874	0.329738
9.0	0.034768	0.096697	0.149092	0.166686	0.322155
10.0	0.026034	0.083398	0.135818	0.153853	0.316245
15.0	0.006372	0.042239	0.091728	0.111111	0.300778
20.0	0.001592	0.022628	0.067003	0.086957	0.295645

Table 8

Q_{loss} characteristics for $M/M/1/(\infty, V) - EPS$ system, $f = 0.5$

V \ ρ	0.2	0.6	0.9	1	2
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	0.911056	0.913478	0.915213	0.915776	0.921042
2.0	0.745578	0.763322	0.775199	0.778917	0.810683
3.0	0.582546	0.625402	0.652807	0.661180	0.728789
4.0	0.445835	0.514064	0.556966	0.569937	0.671755
5.0	0.338092	0.426557	0.482626	0.499579	0.631270
6.0	0.255421	0.357546	0.424058	0.444322	0.601731
7.0	0.192731	0.302454	0.376969	0.399964	0.579671
8.0	0.145425	0.257862	0.338370	0.363625	0.562893
9.0	0.109784	0.221298	0.306188	0.333330	0.549952
10.0	0.082934	0.190971	0.278962	0.307691	0.539863
15.0	0.020586	0.096768	0.188418	0.222222	0.513461
20.0	0.005151	0.051840	0.137631	0.173913	0.504697

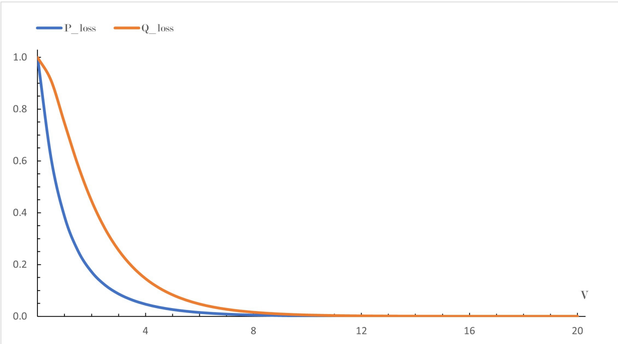

Fig. 11. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=1$, $\rho=0.2$

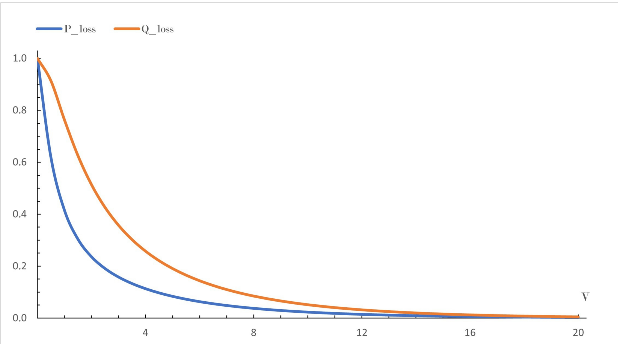

Fig. 12. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=1$, $\rho=0.6$

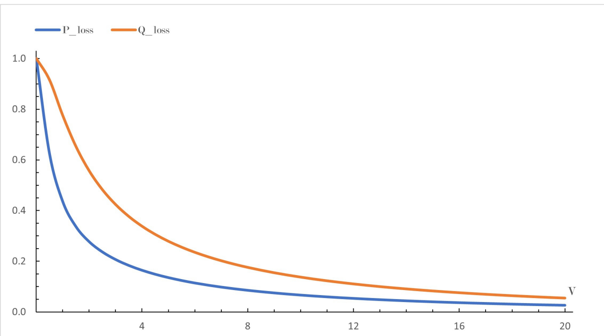

Fig. 13. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=1$, $\rho=0.9$

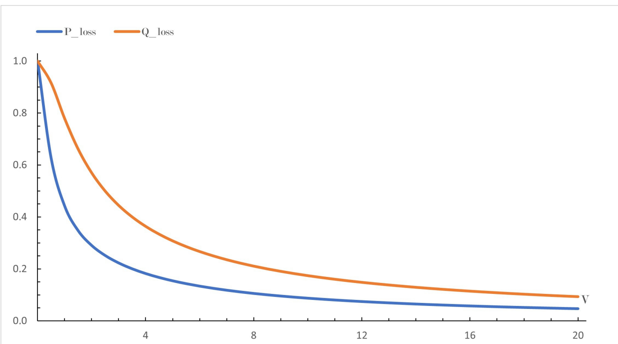

Fig. 14. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=1$, $\rho=1$

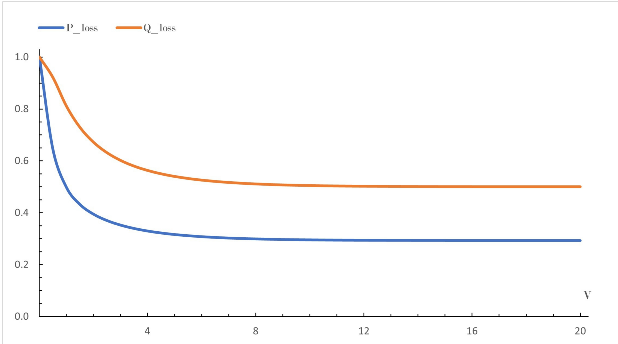

Fig. 15. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=1$, $\rho=2$

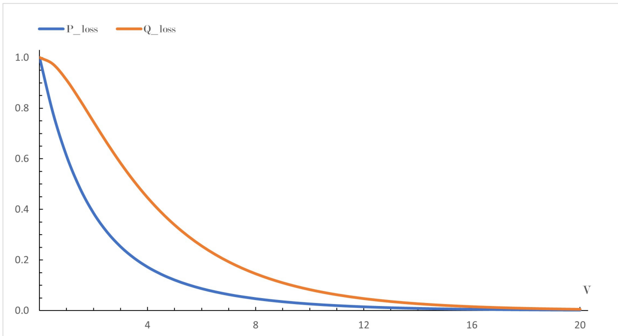

Fig. 16. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=0.2$, $\rho=0.2$

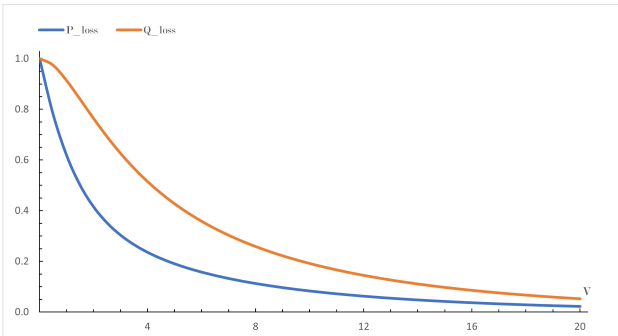

Fig. 17. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=0.2$, $\rho=0.6$

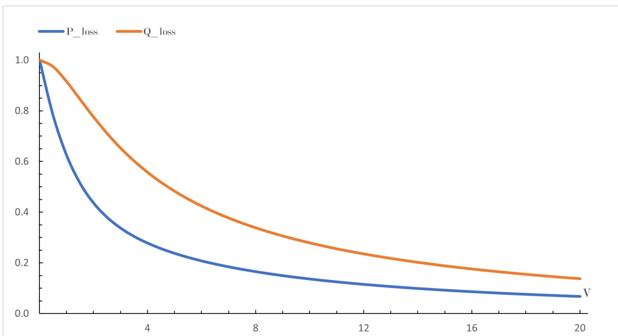

Fig. 18. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=0.2$, $\rho=0.9$

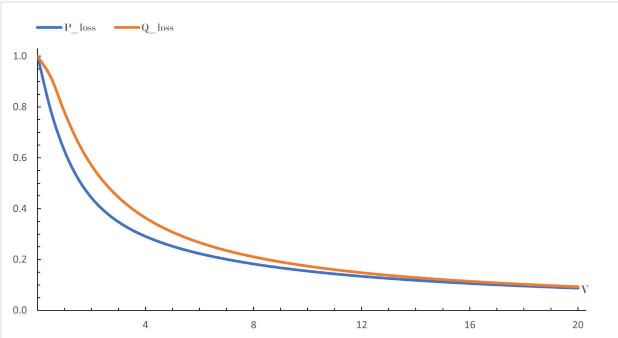

Fig. 19. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=0.2$, $\rho=1$

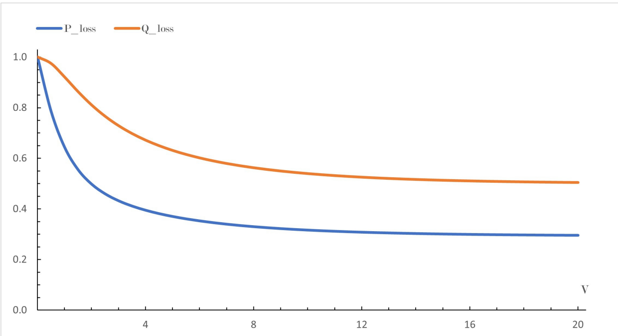

Fig. 20. Loss characteristics for $M/M/1/(\infty,V)-EPS$ queueing system, $f=0.2$, $\rho=2$

4.3. Examples of approximate calculations of loss characteristics

Now we shall analyze two very interesting special cases of queueing systems belonging to the second class (discussed in Section 3).

In the first example, we assume that customer's volume is constant ($\zeta\equiv v_0$, $v_0>0$) and customer's length is proportional to his volume with coefficient c ($c>0$). From the practical point of view, this system can be used to model data packets processing (data packets usually have the same constant size and constant $\frac{1}{c}$ can be understood as the speed of their transmission).

This time we are not able to obtain exact values of loss characteristics (we cannot obtain customers' total volume distribution function in exact form), so we will need previously discussed (also in Section 3) approximations with the use of gamma function (see formulae (55) and (56)). Indeed, if we use formula (52), in this case we obtain:

$$\delta(s)=\frac{1-\rho}{1-e^{-sv_0}\rho}. \tag{57}$$

Obtaining exact form of function $D^{\infty}(x)$ (based on of Laplace transform inversion of function $\frac{\delta(s)}{s}$) is not possible. For this system, we easily obtain:

$$\mathbf{E}\sigma^{\infty}=\frac{\rho v_0}{1-\rho};\qquad \mathbf{E}(\sigma^{\infty})^2=\frac{\rho(1+\rho)v_0^2}{(1-\rho)^2},$$

whence p and g parameters of approximation equal (see formula (56)):

$$p=\frac{1}{\rho};\qquad g=\frac{1-\rho}{\rho v_0}.$$

To calculate estimations of loss characteristics, we now use function $D_*^{\infty}(x)$ defined by formula (55) and the following approximate formulae (based on (26) and (27)):

$$P^*_{\text{loss}}=1-\frac{1}{D^{\infty}_*(V)}\int\limits_0^V D^{\infty}_*(V-x)\delta(x-v_0)\,\mathrm{d}x; \tag{58}$$

$$Q^*_{\text{loss}}=1-\frac{1}{v_0D^{\infty}_*(V)}\int\limits_0^V xD^{\infty}_*(V-x)\delta(x-v_0)\,\mathrm{d}x, \tag{59}$$

where $\delta(x)$ is Dirac-delta function.

It can be easily shown that in this case $P_{\text{loss}}=Q_{\text{loss}}$ because customer's volume is constant. We can also notice that approximation is valid only for $V>v_0$.

Results of calculations for two different values of $\rho<1$ and two values of v_0 are presented in Tables 9–10.

Now we come to the second example. We assume that customer's volume is uniformly distributed on interval $[0,b]$ ($b>0$) and customer's length is proportional to his volume with some positive coefficient c.

Table 9
Loss characteristics for $M/D/1/(\infty,V)-EPS$ system, $v_0=1$

V \ ρ	0.5	0.9
2.0	0.206872	0.300381
3.0	0.114867	0.221742
4.0	0.056366	0.170864
5.0	0.026103	0.135887
6.0	0.011639	0.110599
7.0	0.005047	0.091594
8.0	0.002141	0.076877
9.0	0.000893	0.065210
10.0	0.000367	0.055783

Table 10
Loss characteristics for $M/D/1/(\infty,V)-EPS$ system, $v_0=2$

V \ ρ	0.5	0.9
3.0	0.244055	0.355595
4.0.	0.206872	0.300381
5.0	0.157954	0.256527
6.0	0.114867	0.221742
7.0	0.081223	0.193751
8.0	0.056366	0.170864
9.0	0.038573	0.151866
10.0	0.026103	0.135887

In this case we also are not able to obtain exact values of loss characteristics so we use approximation formulae again. More precisely, formula (52) takes the form:

$$\delta(s) = \frac{1-\rho}{1+\dfrac{2e^{-bs}\rho(1-e^{bs}+bs)}{b^2s^2}}. \tag{60}$$

For this system, we easily obtain:

$$\mathbf{E}\sigma^{\infty} = \frac{2b\rho}{3(1-\rho)}; \quad \mathbf{E}(\sigma^{\infty})^2 = \frac{b^2\rho(9+7\rho)}{18(1-\rho)^2},$$

whereas p and g parameters of approximation equal:

$$p = \frac{8}{1+7\rho}; \quad g = \frac{12(1-\rho)}{b(1+7\rho)}.$$

To calculate estimations of loss characteristics, we now use function $D_*^{\infty}(x)$ defined by formula (55) and the following approximate formulae obtained from (26) and (27):

$$P_{\text{loss}}^* = 1 - \frac{1}{bD_*^{\infty}(V)} \int_0^V D_*^{\infty}(V-x)H(b-x)\,\mathrm{d}x; \tag{61}$$

$$Q_{\text{loss}}^* = 1 - \frac{2}{bD_*^{\infty}(V)} \int_0^V xD_*^{\infty}(V-x)H(b-x)\,\mathrm{d}x, \tag{62}$$

where $H(x)$ is left-sided continuous version of the Heaviside's unitstep function.

5. SUMMARY AND FINAL REMARKS

In the present paper, we have investigated problem of calculating loss characteristics for queueing systems with random volume customers and limited memory buffer. We have shown ways of obtaining exact values of loss characteristics for two classes of queueing systems.

For the first class of systems, in which customer's service time is independent of his volume, we have obtained general results connected with formulae of total customers' volume as well as formulae defining two main loss characteristics: loss probability P_{loss} and probability of losing information unit Q_{loss}. In addition, we have proved that it is possible to obtain the number of customers distribution for such systems if we know analogous distribution for systems with limited total volume when form of customer's volume distribution function is given in exact form.

For the second class of models, in which customer's service time is dependent on his volume, we have analyzed the case of processor sharing system. For this system, we have also presented the formulae defining total customers' volume distribution function and loss characteristics. We have also made some exact and approximate calculations for some special cases of analyzed models, illustrating them with graphs of loss characteristics treated as the functions of the size of memory buffer. Obtained results are very important both from the theoretical and practical point of view. Our calculations can be used during real computer or telecommunication systems design and analysis processes.

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