BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 74(5), 2026, Article number: 1951
DOI: 10.24425/bpasts.2026.1951

The impact of the functional on the quality of the heat transfer coefficient mapping under fourth-kind boundary conditions using swarm algorithms

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Abstract. The article aims to analyze three different functionals used as error norms in reconstructing the heat transfer coefficient κ under fourth-kind boundary conditions. The heat transfer coefficient κ is significant in modeling heat conduction at the interface of the casting and the mold. The study used two optimization algorithms (ABC – Artificial Bee Colony and ACO – Ant Colony Optimization) to assess how individual functionals respond to input data noise and how this affects the quality and stability of the parameter selection process. The studies were conducted for three levels of noise: 0%, 2%, and 5% of the input data and various population sizes: 5, 10, 15, and 20, with a constant number of 6 iterations. In each case, the average value of the functional and the compliance of the determined parameter κ with the reference value were analyzed. The algorithm efficiency was compared based on result stability and data error resistance. The (L_1) functional demonstrated high regularity and stability regardless of computational conditions. ABC and ACO produced consistent solutions, though ACO showed better repeatability. The (L_2) functional, more sensitive to noise, showed significant differences between the algorithms: ABC reacted strongly to noise, leading to a scatter of results, while ACO maintained high mapping quality. The most important differences were observed for the L_∞ functional, which was most susceptible to data noise. In this case, only ACO could maintain the stability and precision of the solutions, indicating its greater resilience in mapping the parameter κ. The analysis shows that the nature of the functional is crucial for the entire optimization process. It is not the algorithm, but the properties of the functional that determine the difficulty of the inverse problem and the scale of the final error. The selection of the functional should be treated as a strategic decision that affects the convergence, stability, and reliability of the solutions obtained. These conclusions are crucial in the context of engineering applications, where the precision and robustness of the computational method directly translate into the quality and safety of the technological process.

Keywords: swarm algorithms; artificial intelligence; functional stability; functional analysis; heat transfer coefficient; thermal conductivity modeling; computational parameter optimization.

1. INTRODUCTION

The inverse heat conduction problem, especially the identification of the heat transfer coefficient (κ) under fourth-kind boundary conditions, is a key issue in modern thermal engineering. This applies, among other things, to casting processes, where heat exchange at the boundary between the casting and the mold must be modeled with high accuracy. In such cases, it is necessary not only to solve the inverse problem but also to ensure its robustness against measurement noise. For this reason, the choice of the functional, which serves as the error norm in the optimization process, becomes particularly important, as it directly affects the stability, accuracy, and convergence of the solution [1]. In this work, the parameter κ denotes the heat transfer coefficient at the casting-mold interface, rather than the thermal conductivity of the material; throughout the manuscript, we consistently use this meaning in the description of the model and the reconstruction of the parameter [2, 3]. In this work, we assume that the parameter κ describes the heat exchange at the casting-mold interface (a fourth-kind boundary condition), and thus pertains to the contact boundary, not the interior of the material. In this context, it couples the heat flux with the temperature difference on both sides of the boundary. In this sense, the reconstruction of κ is not merely an adjustment of the material parameter for the equation in the casting area, but an estimation of the transmission condition parameter, which directly determines the temperature distribution near the contact zone, and consequently the reliability of the technological process model. In the further part of the manuscript, we explicitly write the relationship: measurement data $U_i(t)$ $\to$ model operator $T_i(t,\kappa)$ $\to$ functional $\mathscr{J}(\kappa)$, which allows for a clear interpretation of the impact of norm selection on the conditioning of the inverse problem [4, 5].

In the literature, three classes of functionals are most commonly used: (L_1), (L_2), and (L_∞). Each of them reacts differently to data noise. The functional (L_1) exhibits natural re-

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Manuscript submitted 2026-01-21, revised 2026-05-12, initially accepted for publication 2026-06-16, published in September 2026.

sistance to anomalies and noise in the data, making it particularly useful in engineering applications. The (L_2) functional, although mathematically attractive due to its convexity and simplicity, is more sensitive to measurement errors. On the other hand, (L_∞) minimizes the maximum local error, but can lead to unstable solutions when the input data is noisy [6]. To avoid ambiguity, we emphasize that the differences between the norms L_1, L_2, and L_∞ should be understood as follows: L_1 limits the influence of outlier observations (local errors do not dominate the global criterion), L_2 amplifies the contribution of large deviations by squaring them, whereas L_∞ is a criterion of extremity, where the evaluation is determined by a single worst point in the entire spatiotemporal domain. For this reason, comparing norms is practically equivalent to comparing strategies for robustness against input data disturbances [4, 7].

Swarm algorithms, such as Artificial Bee Colony (ABC) and Ant Colony Optimization (ACO), are increasingly used to minimize these functionals. These algorithms, inspired by the social behaviors of insects, demonstrate high effectiveness in global optimization tasks, even under conditions of computational loads and measurement errors. Research indicates that ACO is characterized by higher repeatability and stability than ABC, especially in optimizing functionals in inverse heat conduction problems [8, 9]. In the context of swarm algorithms, we note that the comparison of "efficiency" should be formulated not only in terms of the number of iterations but also in terms of computational complexity measured by the number of simple model calls (forward solves), as this directly determines the computation time in the Inverse Heat Conduction Problem. Therefore, in the following sections, we will specify how we measure computation time, what stopping criteria we adopt, and what computation limits each run has. Thanks to this, it will be clear whether the differences between ABC and ACO result from the properties of the algorithms, rather than from one of them practically doing more work in a single step [10]. The current state of research on the identification of κ primarily relies on classical approaches: gradient methods (e.g., CGM), iterative "feedback" methods (e.g., secant), and cooling curve fitting in conjunction with simulation. In recent years, numerous studies have been published in which the heat transfer coefficient is determined directly from experimental data and process simulations (various casting technologies, different contact conditions, temporal variability), highlighting that κ is a critical parameter for the quality of temperature and solidification predictions [2, 3, 11]. At the same time, the role of AI methods is increasing: from evolutionary algorithms (e.g., Genetic Algorithms – GA) to "physics-informed" methods that can estimate parameters and boundary conditions from noisy data. Against this background, swarm algorithms are attractive because they do not require the calculation of derivative values and maintain stability with local minima, typical for the coefficient κ in contact conditions [12, 13].

It is worth noting that the latest publications also introduce hybrid approaches that combine population optimisation with classical methods (e.g., QPSO + CGM, where QPSO stands for Quantum-behaved Particle Swarm Optimisation) in the estimation of the heat exchange coefficient for casting processes, which well justifies the need to separate the influence of the algorithm from the influence of the functional [2, 14].

This article analyzes the impact of the choice of functional (error norm) on the quality of the coefficient κ reconstruction using the ABC and ACO algorithms, with the fourth-kind boundary condition. The behavior of functionals with respect to noisy data was evaluated at three levels (0%, 2%, 5%), with different population sizes (5, 10, 15, 20), and a fixed number of iterations. In each configuration, the accuracy of the κ mapping and the stability of the results were analyzed. The results clearly indicate that it is not the algorithm itself, but the functional properties that determine the difficulty of the inverse problem, thereby affecting the solution's final error and reliability.

The aim of the work is not merely a descriptive analysis of the three norms, but a quantitative assessment of the extent to which the choice of the error functional affects the conditioning and reliability of the reconstruction of the boundary parameter κ for the fourth kind boundary condition. In particular, we treat the norm as an element of the identification method (as a mechanism for determining a global error measure based on deviations calculated at grid points (i,t)), rather than a neutral formal detail, because in practice, the choice of functional affects the robustness to disturbances and susceptibility to the dominance of local deviations. This formulated objective aligns with the current research direction, where methods are increasingly compared not only by their fit value but also by the stability of reconstruction and resistance to measurement data disturbances [2, 15].

The methodological novelty of this work lies in the separation of two layers of the heat exchange coefficient for the boundary condition of the fourth kind: the optimisation layer (ABC vs ACO) and the metric layer (selection of the norm L_1, L_2, L_∞). Unlike typical studies where the algorithm is considered the main source of quality improvement, here we demonstrate in a controlled numerical experiment setup that the stability of the reconstruction of the parameter κ is primarily determined by the properties of the functional, and the algorithm reveals its advantages only in functionals that are most sensitive to disturbances. This approach is in line with the trend in contemporary research on the heat transfer coefficient, where the focus shifts from "one best fit" to repeatability, robustness to disturbances, and comparability of methods at the same computational cost [10, 12].

Since swarm algorithms are stochastic, and the results of a single run can be influenced by random initialisation, we introduce a statistical analysis of the results. In each configuration, we report not only the average values but also measures of dispersion (e.g., standard deviation), which serve as a direct criterion for assessing robustness against input data disturbances. This method of presentation allows us to distinguish situations where the algorithm yields a similar average fit quality but significantly differs in repeatability and reliability [7]. The selection of disturbance levels (0–5%) is treated as a controlled test of the method robustness within the typical range for temperature data used in practical heat transfer coefficient procedures, where even minor disturbances can significantly alter the estimation of the boundary parameter. On the other hand, the range of population sizes (5–20) was chosen to encompass variants from very computationally efficient (quick identification) to more exploratory (greater solution diversification). The fixed number of iterations is a deliberately chosen computational constraint here and allows for the comparison of functionals and algorithms under the same computational workload limit [10].

Our work focuses on obtaining results in the context of optimisation methods used in the identification of the parameter κ. In particular, we emphasize the popularity of the gradient-based methods used (e.g., CGM) [2], simple iterative methods applied to cooling curve fitting (e.g., secant/feedback) [11], evolutionary metaheuristics (e.g., GA) [12], and newer physics-informed AI methods [15]. We base the quality comparison on consistent criteria of reconstruction accuracy κ, result stability in multiple runs, and sensitivity to increased levels of input data disturbance. Such selected measures allow for the evaluation not only of the best fit but also of the repeatability and robustness of the method under conditions close to industrial settings. First, we utilise a typical test procedure designed for inverse problems in casting [30, 31], where the authors construct the test procedure by: (i) imposing a known boundary condition, (ii) generating measurements at fixed points (corresponding to thermocouple readings) with controlled perturbation, and (iii) reconstructing the boundary value in the inverse problem, reporting errors in norms and stability measures. Such a procedure is particularly useful because it allows for the comparison of methods with identical input data and the same disturbance scale. Secondly, for the reconstruction of the heat exchange coefficient at the boundary of the area, where κ is estimated based on real temperature curves and compared with classical methods (CGM) and hybrid methods (e.g., QPSO–CGM), given the algorithm parameters and disturbance resistance [2, 14, 32].

Most of the articles considered by the authors dealing with the reconstruction or optimization of thermal parameters can be divided into four groups. The authors' starting point is the inverse heat problem in the theory of casting processes, in which thermophysical parameters, boundary conditions, or initial temperatures are identified from temperature data [16]. The next papers [17, 18] consider inverse Stefan problems, boundary condition reconstruction, and cooling condition identification in continuous casting, which were also included for comparison. The third group of papers [9, 19–23] describes the ABC/ACO and other artificial intelligence-inspired algorithms applied to inverse heat conduction problems. The last group includes works closest to the authors' topic, related to the reconstruction of the heat transfer coefficient [29] or the coefficient κ under the fourth kind boundary condition in the casting-mold system [5, 24–28].

In the study, the authors adopted the following criteria for comparison to determine which function yields the best results:

- accuracy of the reconstruction of the coefficient κ,
- resistance to input data disturbances,
- stability of results in subsequent runs of the algorithm,
- computational cost,
- applicability for the fourth kind boundary condition with imperfect contact between the casting and the mold.

In this context, swarm methods are particularly advantageous when the inverse problem is ill-posed, the error functional exhibits irregular behavior, and the solution to the direct problem is obtained numerically. In such cases, the method does not need to rely on explicit gradient information, which is a practical advantage of ABC and ACO [9, 24, 26]. At the same time, we noted a limitation: swarm methods require many solutions to the direct problem, so for very smooth, well-conditioned problems, deterministic or gradient methods may be computationally faster. This distinction has physical significance. In thermomechanical and casting issues, a small average error does not always indicate a technologically safe solution. The local temperature error near the boundary of the casting mold can be more important than the average error across the entire area [27].

Table 1 presents a comparison of selected works on inverse problems of heat conduction, solidification, and thermal parameter reconstruction. The first group of entries (entries 1–3) concerns classical inverse heat transfer problems in casting. The second group (positions 4–6) covers the application of genetic algorithms and other evolutionary methods to Stefan and solidification problems. The third one (positions 7–9) concerns swarm algorithms in inverse heat conduction problems. And the fourth (positions 10–16), the last one, covers the reconstruction of the coefficient κ under the fourth kind of boundary condition. The novelty lies in the combination of several elements: the reconstruction of the coefficient κ at the boundaries of the casting and the mold, the fourth-kind boundary condition, the ABC and ACO algorithms, and the analysis of how the error functional affects the quality, stability, and robustness of the solution.

A literature review shows that artificial intelligence and metaheuristic methods have already been applied to inverse heat conduction and solidification problems, among others, for reconstructing heat fluxes, cooling conditions, and heat transfer coefficients. Most of the previous works, however, focused on external boundary conditions, the third kind of boundary condition, ideal contact at the interface between the casting and the mold, or on the comparison of optimization algorithms themselves, without an in-depth analysis of the impact of the error functional. The research gap addressed in this work concerns the reconstruction of the coefficient κ describing heat flow through the separating layer between the casting and the mold under a fourth-kind boundary condition, while simultaneously assessing the impact of various error functionals on the accuracy, stability, and robustness of the ABC and ACO swarm algorithms.

In light of the conducted analyses, as well as in the context of industrial applications where the accuracy and robustness of the algorithm have a direct impact on the safety of technological processes, the following research questions become crucial:

- How does the choice of functional (L_1), (L_2), and (L_∞) affect the stability and accuracy of the heat conduction coefficient κ representation in the presence of various levels of noise?
- To what extent do swarm algorithms (ABC and ACO) differ in efficiency in minimizing individual functionals, especially with noisy data?
- Which of the analyzed functionals provides the best compromise between data error resilience and the precision of the mapping κ in the context of real-world engineering applications?

Table 1
Comparison of selected works on inverse problems of heat conduction, solidification, casting processes, and thermal parameter reconstruction using artificial intelligence methods

No.	Work	Considered problem	Method / parameter	Importance for the work
1	Mochnacki *et al.* [16]	Casting processes and inverse heat transfer problems	Numerical modeling; thermal parameters and boundary conditions	Provides the theoretical foundation for inverse heat transfer problems in casting processes.
2	Słota [17]	Three-phase inverse Stefan problem	Genetic algorithm; heat transfer coefficient	It demonstrates the application of an evolutionary algorithm in the inverse problem with phase change.
3	Słota [18]	Solidification of pure metals	inverse Stefan problem; boundary conditions and heat flux	Presents the recreation of boundary conditions during solidification.
4	Hetmaniok *et al.* [9]	Inverse heat conduction problem with a third-kind boundary condition	ABC, ACO, minimization of the temperature error functional, heat transfer coefficient α	Both algorithms effectively reconstructed the parameter and temperature distribution. The work serves as a reference point for the use of ABC/ACO in our reconstruction of κ under the fourth kind boundary condition.
5	Hetmaniok *et al.* [19]	Inverse heat conduction problem	ABC and ACO; thermal parameters	Enables the comparison of two swarm optimization methods.
6	Hetmaniok *et al.* [20]	Solidification of a binary alloy in casting form	FEM and bee algorithm; heat transfer coefficient	It is closely related to inverse modeling of solidification in the mold.
7	Hetmaniok [21]	Inverse problem of alloy solidification	ABC; parameters of inverse problems	It provides a broader context for the application of the ABC algorithm in inverse analysis.
8	Zielonka *et al.* [22]	Continuous casting of steel ingots	Swarm algorithm; cooling conditions	Confirms the usefulness of swarm algorithms in reconstructing casting process conditions.
9	Zielonka *et al.* [23]	Solidification of a binary alloy with shrinkage	Immunological algorithm; heat transfer coefficient	Shows an alternative AI approach to inverse solidification problems.
10	Gawrońska *et al.* [24]	Casting system and casting mold with a conductive layer	ABC and ACO; coefficient κ	Directly pertains to the fourth kind of boundary condition and imperfect thermal contact.
11	Zych *et al.* [25]	Numerical heat conduction model	ABC and ACO; coefficient κ	Analyzes the impact of input parameters on reconstruction accuracy.
12	Gawrońska *et al.* [26]	Casting system and casting mold with an intermediate layer	ABC and ACO; coefficient κ	Confirms the possibility of applying AI algorithms to reconstruct the fourth kind of condition.
13	Dyja *et al.* [27]	Selection of the heat transfer coefficient	ABC and ACO; heat transfer coefficient	Discusses the impact of the location of control points on reconstruction accuracy.
14	Zych *et al.* [28]	Optimization of the temperature field	ABC; coefficient κ	It concerns the selection of the coefficient κ in the heat conduction model.
15	Zych *et al.* [5]	Condition of the fourth kind and different ranges of reference values	ABC and ACO; coefficient κ	It expands the analysis to include different ranges of reference values.
16	Zych [29]	Casting-mold system	Artificial intelligence algorithms; heat transfer coefficient	It constitutes a comprehensive study of the problem of reconstructing the heat transfer coefficient between the casting and the mold.

2. FUNCTIONALS

In inverse heat conduction problems - especially those solved using optimization algorithms – a key element is the proper functional formulation. The functional measures the discrepancy between experimental (measured) data and the results obtained from the numerical model. Minimizing this functional yields the best approximation of the unknown parameter, which could, for example, be the heat transfer coefficient κ.

In further considerations, we adopt the discrete form of functionals, resulting directly from the fact that measurement data are available at a finite number of spatial points and time moments. As a consequence, $\mathscr{J}(\kappa)$ is the sum (or maximum) of local errors on the grid (i,t), and the comparison of norms reduces to comparing the method of aggregating these errors into a single optimisation criterion. Such an approach is crucial because in the heat exchange coefficient, small deviations at selected moments or points can disproportionately affect the reconstruction of the interface parameter [4].

In the context of population-based optimization methods, such as swarm algorithms (e.g., ABC, ACO), the fitness function serves as a direct criterion for evaluating individuals (solutions). For this reason, its structure has a crucial impact on the algorithm behavior, including convergence, stability, robustness to disturbances, and the quality of the final solutions [9].

In this work, we operationally understand stability as the repeatability of the solution, i.e., a small variance in the estimated value of κ across a series of independent runs for the same data configuration and algorithm parameters. We define robustness to disturbances as the maintenance of fit quality (the value of the functional and the error κ relative to the reference value) with an increase in the level of disturbances in the input data. These two measures are mutually exclusive: an algorithm can be stable but consistently burdened by systematic error, or conversely, accurate on average but unstable (high variability in results) [5, 24].

In the literature, three types of error functionals are most commonly used: based on the norm (L_1), (L_2), and (L_∞). Each of them is characterized by different numerical properties and varying sensitivity to disturbances, which translates into their practical usefulness in specific engineering and computational applications.

The (L_1) functional as a measure of mapping error

The norm (L_1) is based on the sum of the absolute differences between the model and the measured temperatures. Its greatest advantage is its resistance to outliers and stability in the presence of disturbances in the input data:

$$\mathscr{J}_{L_1}(\kappa) = \sum_{t=1}^{N_t} \sum_{i=1}^{N} |T_i(t,\kappa) - U_i(t)|, \tag{1}$$

where

- $\mathscr{J}_{L_1}(\kappa)$ – the value of the error functional for a given distribution of the heat transfer coefficient κ,
- $T_i(t,\kappa)$ – the temperature value obtained from the computational model at the measurement point i and at the time t, dependent on the parameter κ,
- $U_i(t)$ – the value of the temperature measured (reference) at point i and time t,
- N – the number of spatial measurement points,
- N_t – the number of time steps.

This functional performs exceptionally well under computational conditions with random disturbances, as confirmed by studies on the inverse heat conduction problem [33, 34].

The L_2 functional as the classical measure of mean squared error

The functional based on the L_2 norm is the root-mean-square error (RMS). It is a classic choice in numerical analysis, mainly due to its strong mathematical properties, such as continuity and differentiability, which favor efficient optimization:

$$\mathscr{J}_{L_2}(\kappa) = \sqrt{\sum_{t=1}^{N_t} \sum_{i=1}^{N} (T_i(t,\kappa) - U_i(t))^2}, \tag{2}$$

where

- $\mathscr{J}_{L_2}(\kappa)$ – the value of the error functional for a given distribution of the heat transfer coefficient κ,
- $T_i(t,\kappa)$ – model temperature at point i and time t, dependent on parameter κ,
- $U_i(t)$ – measured (reference) value at the same point in time,
- N – the number of spatial points,
- N_t – number of time steps.

Although the L_2 norm allows for precise fitting at given points, it also exhibits high sensitivity to even minor noise in the input data [35].

The L_∞ functional as a measure of maximum local error

The L_∞ norm defines the maximum local discrepancy between the simulation and the measurement data. In practice, this means the maximum error occurring over the entire area and time:

$$\mathscr{J}_{L_\infty}(\kappa) = \max_{t=1,\ldots,N_t} \max_{i=1,\ldots,N} |T_i(t,\kappa) - U_i(t)|, \tag{3}$$

where

- $\mathscr{J}_{L_\infty}(\kappa)$ – the value of the maximum mapping error,
- $T_i(t,\kappa)$ – computational temperature at point i and time t,
- $U_i(t)$ – reference temperature at the same point and time,
- N – the number of spatial points,
- N_t – number of time steps.

This functional finds application in system analysis, where controlling errors at critical points is of paramount importance – for example, in assessing material safety or the quality of a technological process. The applications of the L_∞ functional have been extensively discussed, among others [36], who emphasize its usefulness in analyzing the boundary resistance of a model to extreme data.

The choice of the function is not merely a formal matter but a strategic decision that affects all aspects of the inverse process: from the convergence of the algorithm through numerical stability to the interpretability of the results. In combination with swarm algorithms, which are characterized by high flexibility and resistance to local minima, an appropriately selected error measure can significantly enhance the quality and reliability of the obtained parameters.

To maintain full methodological transparency in the manuscript, a complete set of parameters for the ABC and ACO algorithms that were modified during execution, as well as the adopted termination criterion for the calculations, are presented. Such clarification is necessary because, in the case of metaheuristic methods, the evaluation and comparison of algorithm effectiveness without clearly defined parameterisation can lead to ambiguous conclusions. It is particularly important to ensure a comparable computational effort for both methods, understood, for example, as an equal number of model calls, even if the optimisation procedure is conducted with a fixed number of iterations. The evaluation of the quality of the obtained solutions should also be supplemented with an analysis of the result dispersion, which is in line with good practices in reporting research on stochastic algorithms [10, 37].

Due to the nature of the work, which involves the application of stochastic algorithms coupled with a numerical solver, the implementation method is presented in the form of a parameter table along with the range of values analysed in the disturbance study. The adopted solution allows for the reproduction of the procedure regardless of the programming environment used and meets the reporting standards accepted in research on metaheuristics in engineering applications [37].

3. INTRODUCTION TO RESEARCH

In issues related to heat conduction in casting processes, the heat transfer coefficient κ, which determines the intensity of heat exchange between the casting and the mold, remains a critical physical parameter. The accurate reconstruction of this parameter based on measurement data constitutes a classic inverse problem characterized by strong nonlinearity and sensitivity to disturbances in the input data. Particular attention was paid to the boundary between the casting and the mold, where a fourth-kind boundary condition with imperfect contact was applied. In this context, the coefficient κ describes the heat conduction through the layer separating the casting and the mold and thus has a direct physical interpretation [24, 26].

A unique element of the procedure is the coupling of the solver solving the heat conduction equation using the finite element method with the swarm optimization module. Each value of κ proposed by ABC or ACO is automatically introduced into the fourth-kind boundary condition in the solver. Next, the direct problem is solved, the temperatures at the control points are determined, the value of the error functional is calculated, and, based on this, the algorithm updates the population.

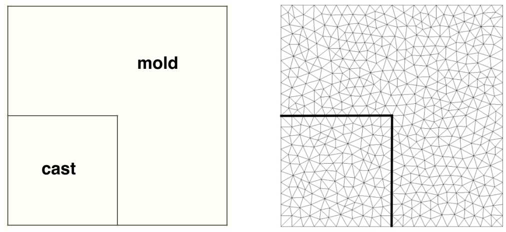
Computational scheme:

1. The geometry was defined in Fig. 1a, the material parameters in Table 2, and the initial condition and the boundary conditions. The initial temperature was $T_0 = 960$ [K] for casting and $T_0 = 590$ [K] for the casting mold, respectively. All simulations were conducted for the alloy Al-2%Cu.
2. A finite element mesh was generated with a clearly defined boundary between the casting and the mold (Fig. 1b).
3. A solution was prepared for the reference value $\kappa = 1000$ [W/m^{2}K].
4. An acceptable range of values for the reconstructed coefficient $\kappa = 900-1500$ [W/m^{2}K] was adopted.
5. An initial population for the classical ABC or ACO algorithm was generated: 5, 10, 15, and 20 individuals.
6. For each proposed value of κ, the direct problem was solved.
7. The temperatures at the control points were determined.
8. The value of the selected error functional was calculated.
9. The population was updated according to the rules of the classical forms of ABC or ACO algorithms.
10. The procedure from steps 6–10 was repeated until the stopping criterion was met, that is, for 6 iterations.
11. The value of κ for which the error functional reached its minimum value was taken as the result.

Table 2
Material properties

	Cast	Casting mold
ρ, kg/m^3	2824	7500
c, J/kgK	1077	620
λ, W/mK	262	40

First, a solution to the direct problem was obtained for the known reference value κ. Next, it was tested whether the inverse procedure could reconstruct this value from synthetic data. In subsequent steps, the calculations were repeated for different population sizes, numbers of iterations, and levels of input data noise. Such an approach is consistent with previous papers, in which the quality of reconstruction was assessed by comparing



(a) (b)

Fig. 1. (a) Geometry, (b) 576-nodes finite element mesh

reference and reconstructed values and by analyzing the impact of data disturbances [24–26, 28].

The numerical experiment for the direct solution was conducted using proprietary software, which was validated and verified experimentally [38].

This work conducted a comparative analysis of three functionals used as measures of mapping error: the L_1, L_2, and L_∞ norms. The study aimed to determine how selecting a specific functional affects the quality of the determined distribution of the parameter κ and its robustness to data disturbances. The efficiency of two optimization algorithms from the swarm methods group was also analyzed: ABC and ACO, with particular attention to the stability and repeatability of the obtained results [39, 40].

The parametric settings associated with the ABC and ACO algorithms were taken from literature items [30, 31]. The main parameters in our calculations, respectively, are:

1) for the ABC lgorithm:
   - food sources – number of bees in the population,
   - search area – the range of the kappa parameter,
   - a number of iterations – the rerun number of food source searching processes by individuals, consistent with the results presented in the calculations.
2) for the ACO algorithm:
   - number of iterations – the rerun number of food source searching processes by individuals, consistent with the results presented in the calculations,
   - the initial number of ant population – the smallest number of individuals as a fixed parameter of the algorithm,
   - the number of ants in the population – the actual number of individuals as an input parameter of the algorithm,
   - Intensification Factor (Selection Pressure) – the value of 0.5, the fixed default parameter of the algorithm,
   - Deviation-Distance Ratio – value 1, the fixed default parameter of the algorithm search area – the range of the kappa parameter.

The experiments were conducted at three levels of input data noise (0%, 1%, and 2%) and for populations of sizes 5, 10, 15, and 20 individuals. In each configuration, the calculations were performed for six iterations and three runs, recording the average values of the functional and the compliance of the determined parameter κ with the reference value. In the conducted calculations, controlled disturbances were introduced to the tem-

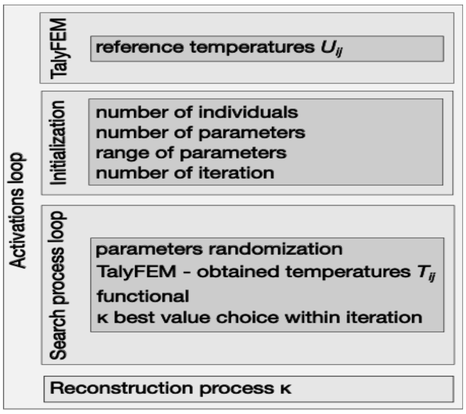


Fig. 2. Scheme of problem solution

perature data to assess the robustness of the identification procedure against measurement errors. In experimental conditions, temperature measurement using thermocouples is usually characterised by good accuracy. In practice, it is assumed that the measurement error is within approximately ±2°C, which corresponds to an uncertainty level not exceeding a few percent and is consistent with the range of up to 5% adopted in the study [24]. Based on experience, we selected three disturbance values: 0, 2, and 5%. At a disturbance level of 5%, we observed that the input parameters of the ABC and ACO algorithms do not guarantee finding the best solution in relation to the reference value.

The geometric models and the finite element mesh were developed in the GMSH program. At the same time, the numerical simulations were conducted using the *TalyFEM* library [41,42], which is based on the finite element method and the data structures of the PETSc library (including solvers, sparse matrices, and vectors). The ABC and ACO algorithms were implemented in Python and integrated with the computational procedures.

The calculations were performed on a 64-bit architecture computer with an Intel Core i9-13900KS processor, 64 GB of DDR5 RAM at 5600 MHz, and the Linux operating system (Ubuntu distribution). The hardware computing platform was selected to ensure reproducibility and numerical efficiency while simultaneously testing a large number of parameter combinations.

4. COMPUTATIONAL RESULTS AND COMPARATIVE ANALYSIS

The analysis of the results presented in Table 3 allows for the assessment of the impact of the choice of functional and the applied optimization algorithm on the quality of the heat transfer coefficient κ mapping, under zero noise level and a constant number of iterations.

The functional L_1, although characterized by higher error values J, demonstrates high stability in the mapping of the parameter κ, regardless of the population size or the type of algorithm. For example, for ACO and 20 individuals, $\kappa = 1000.063$ was obtained with an error of $J = 2.090$, which, despite the greater deviation in the functional value, still indicates a very good parameter mapping. This characteristic of the L_1 norm makes it

Table 3

Results of the parameter κ identification for various functionals and algorithms (0% noise, 6 iterations), Alg – type of the algorithm, Noise – % noise of reference values, N – number of individuals participating in the calculations, κ – value of the parameter chosen by the given algorithm, J – value of the functional, t – computation time given in minutes

Norm	Alg	Noise	Iter	N	κ	J	t
L_1	ABC	0%	6	5	995.788	135.221	8.621
				10	1000.593	73.709	17.626
				15	1000.173	13.310	22.757
				20	999.993	29.802	28.740
L_2	ABC	0%	6	5	1015.604	13.941	10.110
				10	999.695	0.620	17.350
				15	1000.606	7.114	22.946
				20	998.349	1.729	28.424
L_∞	ABC	0%	6	5	994.340	5.229	9.646
				10	1002.082	1.907	17.008
				15	997.546	3.210	22.819
				20	999.582	0.384	28.474
L_1	ACO	0%	6	5	996.789	102.481	1.330
				10	999.848	20.882	2.433
				15	1000.561	18.838	3.567
				20	1000.063	2.090	5.273
L_2	ACO	0%	6	5	997.269	2.520	1.337
				10	1000.312	0.287	2.437
				15	1000.064	0.077	3.913
				20	1000.052	0.072	5.340
L_∞	ACO	0%	6	5	994.394	47.994	1.330
				10	1000.440	3.487	2.433
				15	999.961	0.386	5.367
				20	999.956	0.481	5.155

particularly useful for noisy data, where reconstruction stability is crucial.

The functional based on the L_2 norm achieves the highest fitting precision, with a greater impact on solution stability and accuracy, especially with larger populations. For the ACO algorithm and 20 individuals, a $\kappa = 1000.052$ value was obtained with a very low error of $J = 0.072$, indicating an almost ideal parameter reconstruction. ABC also achieves high L_2 fitting quality, but at a significantly longer computation time, which may be a limitation in more complex analyses.

On the other hand, the L_∞ functional, which maximizes the local mapping error, turns out to be the most sensitive to the algorithm parameterization. For small populations, it generates relatively high errors (e.g., $J = 47.994$ for ACO and 5 individuals), but with an appropriate increase in the number of individuals, its effectiveness significantly increases; for 20 individuals, $\kappa = 999.956$ and $J = 0.481$ were achieved, which is already a level comparable to the L_2 functional. Nevertheless, the high variability in the results indicates lower stability for this approach.

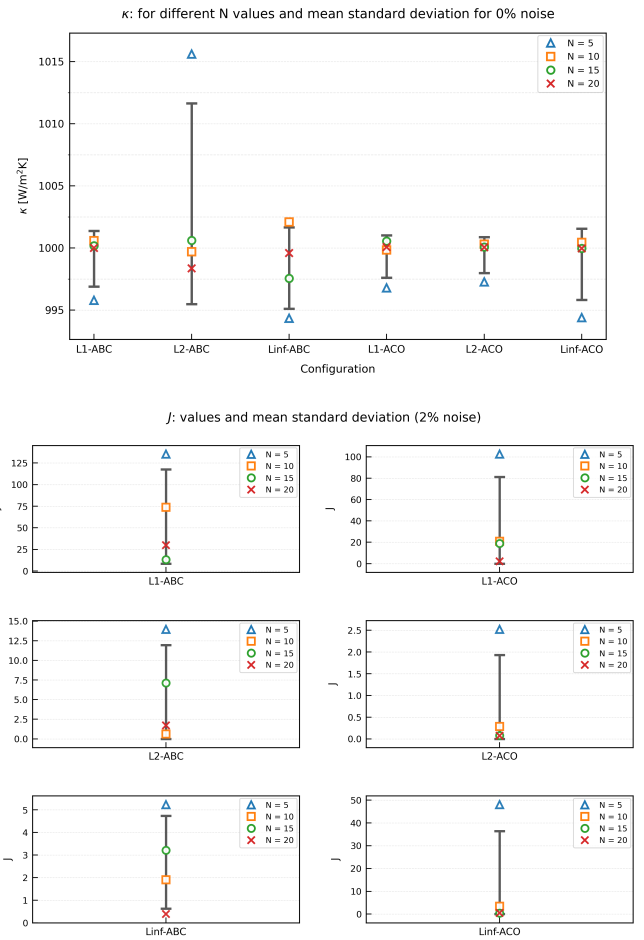


Fig. 3. Standard deviations for kappa and the functional relative to the number of individuals for data in Table 3

In comparing the efficiency of swarm algorithms, ACO advantage in computation time is noticeable. ACO achieves comparable or better κ values and lower J functional values for all three functionals, with significantly shorter computation times than ABC. These differences are particularly significant for larger populations, where optimization time becomes a key practical factor.

In summary, the L_2 functional guarantees the highest accuracy in mapping the parameter κ, which is beneficial with well-filtered data. The L_1 functional, on the other hand, represents a compromise between precision and robustness to noise and can be particularly useful in noisy measurement conditions. Although the L_∞ functional is most sensitive to individual deviations, it can identify local extreme errors that can be significant in engineering applications related to system safety. On the other hand, the ACO algorithm is more computationally efficient than ABC, especially with larger populations, offering a good balance between solution quality and time to obtain it.

Analyzing the data in Table 4, one can observe a clear impact of the functional on the stability and accuracy of identifying the parameter κ under 2% disturbance conditions. The L_1 functional proved to be the least robust to the presence of disturbances, which is reflected in the high values of the error functional J for the ABC algorithm (e.g., $J = 25050.492$ with 5 individuals). These values are even an order of magnitude higher than in the case of the L_2 or L_∞ norms. Although the value of κ remains close to the expected value, the high mapping error indicates a strong susceptibility of L_1 to outliers in this scenario.

Table 4

Results of the parameter κ identification for various functionals and algorithms (2% noise, 6 iterations), Alg – type of the algorithm, Noise – % noise of reference values, N – number of individuals participating in the calculations, κ – value of the parameter chosen by the given algorithm, J – value of the functional, t – computation time given in minutes

Norm	Alg	Noise	Iter	N	κ	J	t
L_1	ABC	2%	6	5	1069.497	25050.492	9.453
				10	1005.435	24533.439	16.384
				15	1004.848	24533.201	23.642
				20	1001.579	24532.288	29.877
L_2	ABC	2%	6	5	987.605	366.639	9.519
				10	1007.086	365.800	16.292
				15	1002.149	365.784	22.152
				20	1009.949	365.800	30.736
L_∞	ABC	2%	6	5	1007.953	365.785	9.457
				10	1008.217	365.790	16.536
				15	1003.698	365.771	23.217
				20	1002.075	365.775	29.280
L_1	ACO	2%	6	5	999.974	24532.708	1.317
				10	1000.556	24532.444	2.440
				15	1001.953	24532.273	5.123
				20	1001.953	24532.273	5.123
L_2	ACO	2%	6	5	1014.639	365.888	1.313
				10	1004.003	365.759	2.443
				15	1004.821	365.759	4.000
				20	1004.608	365.758	5.123
L_∞	ACO	2%	6	5	1001.038	7012.762	1.310
				10	1004.271	7010.944	2.440
				15	1004.282	7010.944	3.863
				20	1003.811	7010.940	5.117

On the other hand, the L_2 and L_∞ functionals demonstrated high resistance to noise—the error values J were very stable and almost unchanged regardless of the number of individuals. For example, for L_2 and the ABC algorithm, J oscillates around the value of 365.8, and the obtained values of κ fall within the range of 987.6–1009.9, which confirms the stability of this norm even with minor noises. A similar situation applies to the L_∞ norm, although with slightly higher error values $J \approx 7010.9$ in the case of the ACO algorithm.

In the context of swarm algorithm efficiency, it should be emphasized that ACO typically has significantly lower computational time than ABC. Even with a larger number of individuals (20), the computation times for ACO did not exceed 5.2 minutes, while those for ABC reached up to 30 minutes. At the same time, the mapping quality was comparable and, for some

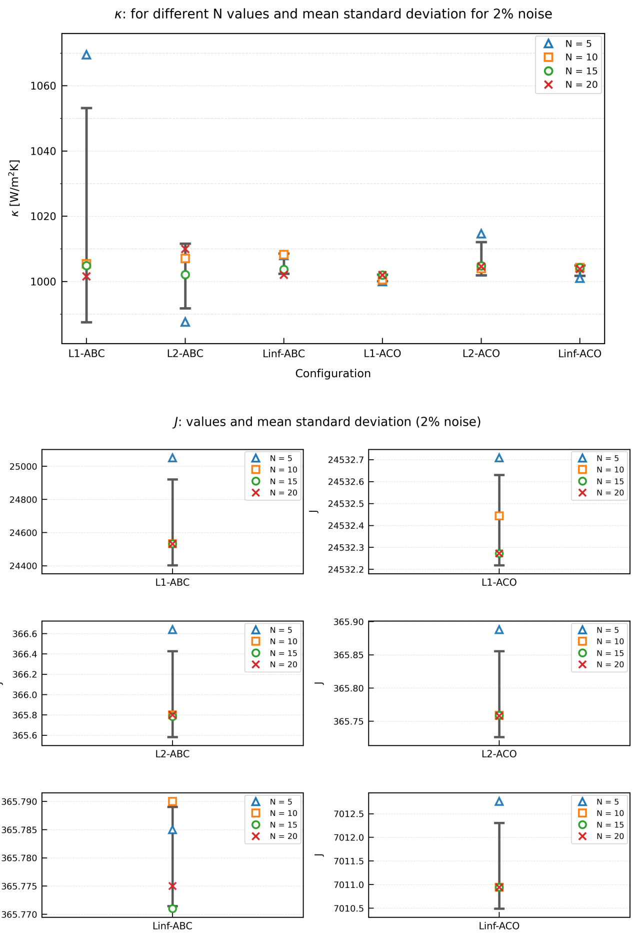


Fig. 4. Standard deviations for kappa and the functional relative to the number of individuals for data in Table 4

Table 5

Results of the parameter κ identification for various functionals and algorithms (5% noise, 6 iterations), Alg – type of the algorithm, Noise – % noise of reference values, N – number of individuals participating in the calculations, κ – value of the parameter chosen by the given algorithm, J – value of the functional, t – computation time given in minutes

Norm	Alg	Noise	Iter	N	κ	J	t
L_1	ABC	5%	6	5	954.343	61089.079	9.122
				10	955.292	61085.671	15.993
				15	956.531	61085.528	20.179
				20	956.836	61085.590	27.177
L_2	ABC	5%	6	5	968.691	909.131	9.066
				10	973.372	909.134	15.448
				15	970.897	909.107	19.900
				20	970.554	909.102	27.490
L_∞	ABC	5%	6	5	974.096	909.141	9.146
				10	969.993	909.101	16.362
				15	970.608	909.101	20.822
				20	970.437	909.100	27.223
L_1	ACO	5%	6	5	947.154	61087.920	1.310
				10	954.734	61085.212	2.400
				15	954.626	61085.195	3.490
				20	954.616	61085.194	4.723
L_2	ACO	5%	6	5	993.504	909.780	1.310
				10	970.946	909.100	2.403
				15	970.568	909.100	3.490
				20	970.533	909.100	4.723
L_∞	ACO	5%	6	5	977.557	17551.717	1.310
				10	976.648	17551.611	2.400
				15	977.046	17551.609	3.490
				20	977.031	17551.608	4.723

functionals, even better. For example, when using the L_2 functional, the ACO algorithm achieved nearly identical error values to ABC while incurring significantly lower computational costs.

Regarding the trade-off between error resistance and the precision of the κ mapping, the most balanced approach is the L_2 functional. It maintains low J error values for ABC and ACO while simultaneously providing estimates of κ close to the expected values. The L_1 functional, despite its good performance on noise-free data, loses stability even with minor disturbances. On the other hand, L_∞, although it offers high robustness and detects maximum local deviations, can generate inflated global error values, as seen in the case of ACO (e.g., $J = 7012.944$).

In summary, the best choice for engineering applications where noise in measurement data is unavoidable is the L_2 norm combined with the ACO algorithm. Such a configuration ensures stability and high computational efficiency, making it an attractive solution for practical implementations of heat-conduction coefficient identification under fourth-kind boundary conditions.

The analysis of the results presented in Table 5 indicates a significant impact of both the choice of functional and the optimization algorithm on the quality of the heat conduction coefficient κ mapping in the presence of 5% noise.

In the case of the L_1 functional, there is a noticeable sensitivity to the presence of noise — the values of the J functional reach very high levels on the order of $6.1 \cdot 10^4$, regardless of the number of individuals and the algorithm used (both ABC and ACO). Despite the high error values, the identified κ values for a larger number of individuals (e.g., 15 or 20) stabilize within ranges close to the reference values, suggesting that L_1 exhibits resistance to single extreme values but loses effectiveness under dispersed noise conditions.

The L_2 functional, based on the mean square norm, exhibits distinctly better stability and lower error values of J – on the order of approximately 909. Both the ABC and ACO algorithms achieve similar κ values, with ACO requiring significantly less computational time to achieve a comparable result. The minimal variability of the value J as the number of individuals increases is worth noting, which may indicate good convergence of the optimization using the L_2 functional under disturbed conditions.

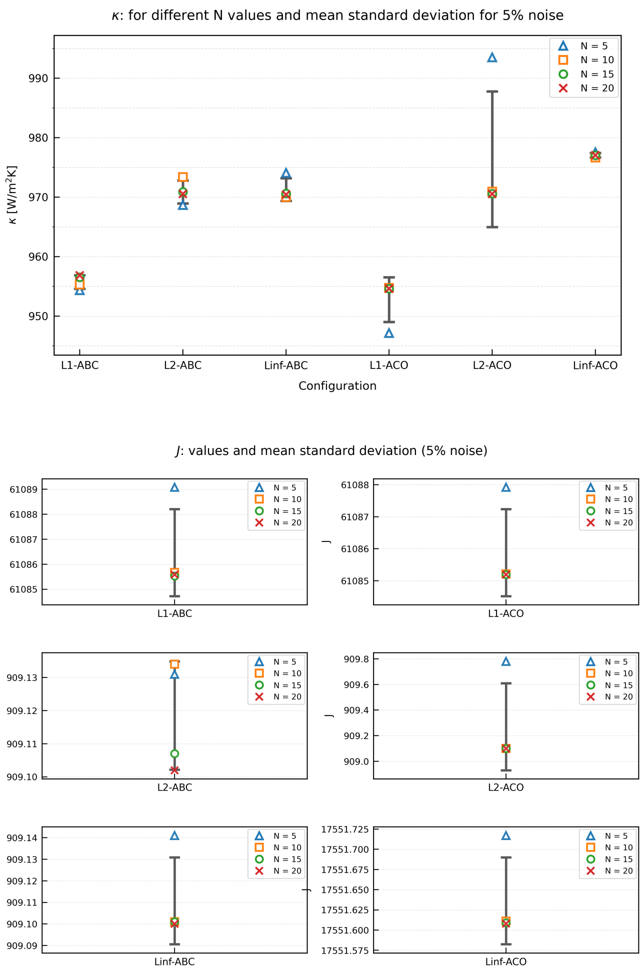


Fig. 5. Standard deviations for kappa and the functional relative to the number of individuals for data in Table 5

An even more precise balance between accuracy and noise resistance can be observed for the L_∞ functional. The error values J for the ABC algorithm are almost identical to those obtained for L_2; however, when using ACO, the L_∞ functional leads to significantly higher error values – on the order of $1.75 \cdot 10^4$. The result indicates that, for the ACO algorithm, the maximum norm reacts more strongly to individual measurement errors and can lead to significant metric distortions when extreme values appear in the data.

Comparing the two swarm algorithms, it is clear that ACO has significantly shorter computation time while maintaining similar or even better accuracy in identifying the coefficient κ, especially for the L_2 functional. The ABC algorithm requires significantly more time with more individuals, which can be a significant limitation in practical applications.

In summary, the L_2 functional is the most universal and stable under 5% noise conditions, offering a good compromise between reproduction accuracy and noise resistance in the data. Combined with the ACO algorithm, it can constitute an effective and computationally efficient solution for inverse identification tasks of heat conduction parameters under fourth-kind boundary conditions.

5. DISCUSSION OF THE OBTAINED RESEARCH RESULTS

The selection of the functional has a crucial impact on the accuracy and stability of the parameter κ mapping. The functional based on the L_2 norm proved to be the most robust to disturbances and provided the best fit both with undisturbed data and in the presence of disturbances in the 2–5% range. The error values J were very low and stable for the ABC and ACO algorithms, and the mapped values κ were close to the reference values, regardless of the number of individuals.

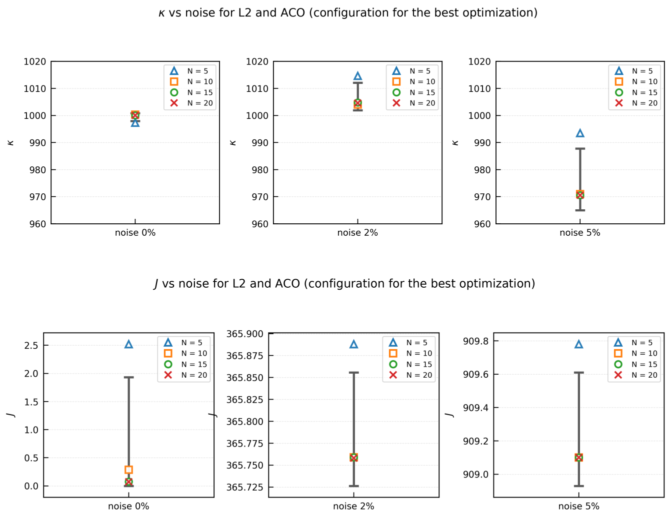


Fig. 6. Standard deviations for kappa and the functional relative to the number of individuals for the L2 norm ACO (different noise levels)

The functional L_1, although effective with noise-free data, showed greater sensitivity to noise, particularly under 2% and 5% disturbance conditions. A sharp increase in the value of the functional J was recorded, even by several orders of magnitude, compared to the other norms. Nevertheless, the value of the parameter κ remained relatively stable, suggesting that L_1 may retain the ability to capture the trend, albeit with significantly less precision in fitting the inverse function.

The L_∞ functional, which emphasizes maximum local deviations, exhibited varied effectiveness. With the ABC algorithm, its behavior was similar to L_2; however, in the case of ACO, the values of J reached very high levels, which may be due to this norm's strong reaction to single deviations in disturbed data. This means that the maximum norm helps detect point errors but does not necessarily guarantee the best global reconstruction.

From a computational efficiency perspective, the ACO algorithm clearly outperforms ABC. For each functional, the computation time was significantly shorter while maintaining comparable, and sometimes even better, mapping quality. These differences were particularly evident with larger numbers of individuals, where ABC reached up to 30 minutes, while ACO was completed in about 5 minutes.

The choice of the functional significantly affects the quality of the parameter κ mapping. L_2 provides the highest precision and stability, and L_1 shows high resistance to extreme values but loses quality with dispersed noise. At the same time, L_∞ allows for identifying local errors but is sensitive to single disturbances.

The differences between swarm algorithms are apparent. ACO proved to be more time-efficient and more stable in the presence of noise, achieving results as good as or better than ABC with significantly shorter optimization time.

The best compromise between error resilience and mapping accuracy κ is provided by the L_2 functional in combination with the ACO algorithm. Such a configuration guarantees high precision, low errors, and computational efficiency, making it an optimal choice for real-world engineering applications where measurement data may be subject to noise.

The results indicate that combining the L_2 functional with the ACO algorithm constitutes the most balanced and practical parameter identification strategy in inverse heat conduction problems.

6. CONCLUSIONS

The conducted analysis of the results of identifying the heat conduction parameter κ under fourth-kind boundary conditions, using the functionals L_1, L_2, and L_∞ and two swarm algorithms (ABC and ACO), allows for the formulation of unequivocal conclusions regarding the effectiveness of these methods under various levels of noise. The absence of standardized benchmarking frameworks represents a critical gap in the literature that demands immediate attention through systematic comparative research. Future studies should establish comprehensive evaluation protocols that simultaneously assess novel methodologies against distinct categories: classical approaches or contemporary engineering methods (including artificial intelligence methods). These comparative studies must move beyond isolated performance metrics by developing unified assessment criteria encompassing accuracy, computational efficiency, robustness to noise and uncertainty, scalability across problem dimensions, interpretability of solutions, and practical applicability in real-world engineering contexts. Researchers should prioritize creating open-access datasets and test cases, derived from both synthetic problems with known ground truth and real engineering applications, that allow reproducible, side-by-side evaluation of diverse methods under identical conditions. Establishing such standards would not only facilitate objective comparison of proposed innovations but also accelerate methodological advancement by enabling the community to identify which approach types excel in specific problem classes (e.g., ill-posed problems, high-dimensional inverse problems, or multi-parameter estimation).

The aim of the study was to determine the extent to which the choice of error functional (L_1, L_2, L_∞) affects the accuracy and stability of the identification of the boundary parameter κ occurring in the fourth kind boundary condition, and to what extent this assessment depends on the optimization algorithm used (ABC vs ACO). The obtained results show that the reliability of the reconstruction is primarily determined by the properties of the functional: changing the norm alters the sensitivity to data disturbances and the stability of the mapping of κ, while the role of the algorithm mainly reveals itself in the repeatability of solutions and computational efficiency within the given functional.

APPENDIX

Comparison of results obtained using ABC and ACO algorithms with methods previously used in inverse heat conduction problems, solidification modeling, and casting process analysis. Classical methods, genetic algorithms, immunological algorithms, Bayesian approaches, and prior applications of swarm algorithms were considered [9, 16–23]. Particular attention was paid to works closest to the subject of the article, namely the reconstruction of the heat transfer coefficient or the coefficient κ under the fourth kind boundary condition in the casting system and the casting mold [5, 24–29].

Based on the results, the authors identify the aspects in which the ABC and ACO methods are advantageous, where they yield comparable results, and where they have limitations. The comparison was conducted according to the following criteria: accuracy of the reconstruction of the coefficient κ, resistance to input data disturbances, stability of results in subsequent runs of the algorithm, computational cost, lack of necessity to determine t he functional gradient, and applicability to the fourth kind

Table 6
Comparison of methods described in the literature with the results obtained in the work for coefficient reconstruction κ

Criterion	Solutions described in the literature	Solutions described in the paper
Accuracy of coefficient reconstruction κ	In the cited literature, both classical methods and AI methods were used. Classical methods, including gradient methods, iterative methods, and cooling curve fitting, were effective in reconstructing heat flux, IHTC, and thermal parameters, but usually required a well-defined model, regularization, or an appropriately chosen starting point [11,17]. Evolutionary and swarm algorithms, such as GA, ABC, ACO, and PSO, also yielded good results in nonlinear and inverse problems [17,20,22,31]. In the works concerning ABC and ACO, very good approximations of the temperature distribution were obtained, and the reconstruction errors were smaller than the input data errors for noise of 1-2% and close to the disturbance level at 5% [9]. Hybrid methods, such as QPSO-CGM, improved accuracy by combining global search with local refinement, yielding correlation coefficients above 0.98 and relative heat-flow errors below 15% [14].	In the analyzed cases, the best reconstruction accuracy κ was achieved for the functional based on the L_2 norm. For both algorithms, ABC and ACO, the κ values remained close to the reference values for both unperturbed data and data with 2–5% perturbations. In this aspect, the results are comparable to earlier AI methods and population methods described in the literature. However, no clear accuracy advantage over all classical or hybrid methods has been demonstrated. The advantage of our approach lies rather in achieving good accuracy without the need to determine the functional gradient. The L_1 functional was effective mainly for noise-free data, while L_∞ indicated local errors well but was weaker as a criterion for global fitting.

Table 6 [cont.]
Comparison of methods described in the literature with the results obtained in the work for coefficient reconstruction κ

Criterion	Solutions described in the literature	Solutions described in the paper
Resilience to input data noise	In the cited works, robustness to noise was mainly analyzed for population-based methods such as GA, ABC, ACO, PSO, and other metaheuristics. In the case of ABC and ACO, it was shown that with noise of 1% and 2%, the reconstruction errors were smaller than the input data error level, and with a 5% disturbance, they were close to the disturbance level [9]. Hybrid methods, such as QPSO-CGM, also performed effectively under measurement disturbance and complex boundary conditions [14]. Classical methods can be accurate, but with disturbed data, they often require regularization, filtering, or additional stability control [4, 17]. Physics-informed approaches are promising with incomplete and noisy data, but their effectiveness depends on the quality of learning, the choice of loss function, and the training cost [13, 15].	The greatest resilience to noise in the studied cases was demonstrated by the L_2 functional, which maintained good fitting quality for data disturbed by 2% and 5%. In this regard, our approach is more advantageous than classical methods, which are sensitive to noise, and is comparable to earlier population methods. The L_1 functional was weaker with distributed disturbances: with noise of 2-5%, the value of J increased by several orders of magnitude, even though the value of κ remained relatively stable. This means that L_1 retained the ability to capture the trend, but provided poorer fitting precision. The L_∞ functional reacted strongly to individual deviations, especially in combination with ACO. It was useful for detecting point errors, but weaker for assessing the global reconstruction. The L_∞ functional reacted strongly to individual deviations, especially in combination with ACO. It was useful for detecting point errors but weaker for assessing global reconstruction.
Stability of results in subsequent runs of the algorithm	In the cited literature, population-based methods such as GA, ABC, ACO, and PSO have been used, among other reasons, because they search multiple points in the solution space and can reduce the risk of getting stuck in a local minimum [31, 39]. However, it has been shown that their results may depend on the number of individuals, the number of iterations, and the control parameters. For the genetic algorithm in the inverse Stefan problem, a small variation in results was observed across different settings of the pseudorandom number generator, with improvements over the Nelder-Mead method [17]. In studies on ABC and ACO, it was also emphasized that both methods work, but ACO can achieve a satisfactory reconstruction with fewer iterations [9].	The stability of the reconstruction primarily depended on the choice of the functional. For the L_2 norm, low and stable values of J and small deviations of the κ values were obtained, regardless of the number of individuals. In this aspect, the results are consistent with the advantages of population methods described in the literature and can be considered comparable to earlier AI approaches. For L_1, the stability of the parameter κ itself was maintained, but the fit quality deteriorated with disturbances. For L_∞, the results were more varied, especially for ACO with disturbed data. This means that L_2 was the best in terms of global stability, while L_∞ was a weaker criterion for reconstructing the entire field.
Computational cost	Classical methods can be more computationally advantageous if the functional changes regularly and predictably, and the model is well-conditioned. In reality, however, inverse problems are usually ill-posed, which affects the ability to obtain accurate results. Moreover, here, artificial intelligence algorithms come to the rescue. Swarm and evolutionary methods require many functional evaluations, which increases the cost, as each evaluation involves solving the direct problem [10,31]. However, the cited literature indicates that ACO may be more advantageous than ABC. In the analyzed inverse heat conduction problem, ACO required about half as many iterations as ABC to achieve a satisfactory approximation [9]. Hybrid methods, such as QPSO-CGM, can reduce costs by combining global search with local refinement [14]. In Bayesian optimization, a similar accuracy to GA was achieved, but the computation time was three to four times shorter [7].	Compared to classical methods, swarm algorithms may be computationally more expensive, as they require many evaluations of the functional and multiple resolutions of the direct problem. However, they gain overall in the balance between the time spent on parameter tuning and the time taken to obtain a solution. In terms of computation time, ACO proved to be clearly more advantageous than ABC. For all analyzed functionals, ACO achieved comparable, and in some cases better, reconstruction quality with shorter optimization time. The differences were particularly noticeable with a larger number of individuals: ABC calculations took up to 30 minutes, while ACO completed them in about 5 minutes. In this regard, ACO performed better than ABC and was consistent with previous literature, indicating that ACO may require fewer iterations.
No need to determine the functional gradient	Gradient methods, conjugate methods, and some classical inverse procedures require information about the functional derivatives or an additional sensitivity model [11, 14]. The cited literature indicates that CGM can fit the temperature well. However, it is dependent on the initial approximation and performs worse with complex or multimodal boundary conditions [14]. AI algorithms, including ABC, ACO, GA, and PSO, operate based on the values of the objective function, making them simpler to apply to irregular, disturbed, or difficult-to-differentiate functions [31, 39].	In this work, this advantage of ABC and ACO was utilized because the reconstruction of κ was carried out through direct minimization of the error functionals L_1, L_2, and L_∞. It was not necessary to determine the functional gradient or build a conjugate model. In this regard, the applied approach is superior to gradient and coupled methods, especially for problems with a fourth-kind boundary condition and imperfect thermal contact. At the same time, this advantage is practical rather than absolute: it does not automatically imply lower computational cost or greater accuracy compared to hybrid methods or well-chosen classical methods.
Applicability for the fourth kind of boundary condition with imperfect contact between the casting and the mold	The closest literature pertains to the reconstruction of the heat transfer coefficient or IHTC at the boundary between the casting and the mold. Classical, iterative, gradient, and hybrid methods have been effectively used to identify thermal contact conditions, but they often require precise measurements, regularization, or a good understanding of the model [2, 3, 5, 11, 14, 24, 26]. In the literature concerning the fourth kind of boundary condition, it has been indicated that the variability of the air gap and measurement difficulties justify the use of methods robust to disturbances and data discontinuities [5].	The obtained results confirm the practical applicability of swarm algorithms for identifying the parameter κ present in the fourth-kind boundary condition. The best compromise between accuracy, disturbance resistance, and computation time was achieved by combining the L_2 functional with the ACO algorithm. Such a configuration ensured low errors, stable κ values, and short computation times. In this aspect, ACO with the L_2 function was more advantageous than ABC. Compared to classical methods, the swarm approach is simpler to apply because it does not require a gradient and is well-suited to the irregular contact problem. Its limitation, like other metaheuristics, is the greater number of functional evaluations, which is why, in terms of cost, it may be weaker than well-tuned classical methods.

boundary condition with imperfect contact between the casting and the mold.

This distinction has physical significance. In casting, a small average error does not always indicate a solution is sufficiently good from a technological perspective. The local temperature error near the boundary between the casting and the mold can be more important than the average error over the entire area. Therefore, in the revised version of the article, it is explained that the choice of error functional is part of the definition of the inverse problem and affects the stability and robustness of the reconstruction of the parameter κ under the fourth kind boundary condition.

In the analyzed cases, the best compromise between reconstruction accuracy, disturbance resistance, and computation time was achieved for the L_2 functional combined with the ACO algorithm. The L_1 functional allowed for the stability of the κ value, but its fitting quality deteriorated with disturbed data. The L_∞ functional was useful for detecting local deviations, but it was more sensitive to individual disturbed points. The L_∞ functional was useful for detecting local deviations, but it showed greater sensitivity to individual disturbed points.

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