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On Stability of Descriptor Fractional Linear Control Systems with Regular Pencils

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Abstract. This paper presents novel stability results for descriptor fractional proportional linear control systems governed by generalized Caputo derivatives. Classical stability notions are extended to fractional settings by introducing Mittag–Leffler stability and asymptotic stability conditions. The analysis distinguishes between nonsingular and singular cases of the system matrix F. In the nonsingular case, explicit solutions are obtained via the generalized Laplace transform, while in the singular case, under regularity of the matrix pencil, a decomposition method is applied. Stability criteria are expressed in terms of eigenvalues of the system matrix G, fractional parameters θ and ρ, and the placement of poles in the complex plane. Examples illustrate the theoretical results, providing a rigorous framework for fractional-order control system analysis.

Key words: Fractional Operator; Descriptor System; Regular pencil; Stability.

1. INTRODUCTION

Stability is a fundamental concept in control theory, ensuring that a system returns to its equilibrium state after disturbances [1]. While stability theory for integer-order systems is well established, fractional-order systems exhibit more complex dynamics due to memory effects, making stability analysis both challenging and essential. The Mittag–Leffler function has emerged as a key tool for characterizing stability in fractional-order systems. Recent studies have investigated various stability properties of fractional-order systems. Finite-time stability of fractional nonlinear systems was examined in [2], while controllability and Hyers–Ulam stability for delayed fractional systems with ψ-Caputo derivatives were studied in [3]. Lyapunov-based stability results for fractional-order neural networks with time-varying delays were proposed in [4]. Ulam-type stability for fractional partial differential equations was analyzed in [5], and a comprehensive survey of stability methods for dynamical systems was presented in [6]. Foundational stability results using the Riemann-Liouville derivative were established in [7], while connections between fractional-order and classical linear time-invariant systems were explored in [8].

In parallel, several related contributions have appeared in the literature (see, e.g., [9]), presenting new sufficient conditions for the asymptotic stability of positive descriptor continuous-time and discrete-time linear systems. Due to strong cross-coupling effects, the Two Rotor Aerodynamical System (TRAS) is commonly controlled using multiple PID and FOPID controllers, with studies reporting improved performance of FOPID schemes [10].

Descriptor fractional linear systems pose additional challenges due to algebraic constraints and parameter uncertainties [11]. Robust stability and stabilization conditions using linear matrix inequalities and feedback control were developed in [12, 13, 14] Robust model predictive control and Lyapunov-based stability analysis were investigated in [15], while unified robust control strategies and qualitative stability analyses were reported in [16, 17]. Despite these advances, the stability of generalized Caputo fractional proportional linear time-invariant systems and descriptor Caputo proportional fractional linear control systems remains insufficiently explored. This paper addresses this gap by deriving analytical stability results using generalized Laplace transforms and the Weierstrass regular pencil decomposition, thereby extending classical stability theory to the generalized fractional-order framework. Section 2. presents essential preliminaries on fractional calculus and generalized Laplace transforms. Section 3. derives the solution of the descriptor generalized Caputo proportional fractional control system and analyzes the stability of both generalized and descriptor systems.

2. PRELIMINARIES

Jarad et al. [18] introduced the concept of generalized Caputo proportional fractional derivative, which applies to a function in relation to another function.

Definition 1. *[18] Let $\rho \in (0,1]$ and $\theta \in \mathbb{C}$ with $Re(\theta) \geq 0$. The Caputo-type left fractional derivative of order ρ, starting at b, is defined as*

$$\begin{aligned}\left({}^{c}D^{\theta,\rho,g}h\right)(\upsilon) &= {}_{b}I^{m-\theta,\rho,g}\left(D^{m,\rho,g}h\right)(\upsilon)\\ &= \frac{1}{\rho^{m-\theta}\Gamma(m-\theta)}\int_{b}^{\upsilon} e^{\frac{\rho-1}{\rho}[g(\upsilon)-g(\tau)]}\\ &\times [g(\upsilon)-g(\tau)]^{m-\theta-1}\left(D^{m,\rho,g}h\right)(\tau)g'(\tau)\,d\tau.\end{aligned}$$

where

$$\begin{aligned}m &= \lfloor Re(\theta)\rfloor + 1,\\ D^{m,\rho,g} &= \underbrace{D^{\rho,g}D^{\rho,g}\cdots D^{\rho,g}}_{m\ times}.\end{aligned}$$

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and

$$(D^{\rho,g}h)(\tau) = (1-\rho)h(\tau) + \rho\frac{h'(\tau)}{g'(\tau)}.$$

Here, θ denotes the fractional order of the derivative, whereas ρ represents the proportional scale factor of the generalized operator.

Jarad et al. [19] defined the generalized Laplace transform for real-valued functions. Bukhsh et al. [20] generalized the concept of generalized Laplace transform to vector-valued functions under real-valued weight function.

Definition 2. *[20]Let $h:[b,\infty)\to\mathbb{R}^n$ be a vector-valued function and $g:[b,\infty)\to\mathbb{R}$ be a continuous, strictly increasing function on $[b,\infty)$. The generalized Laplace transform of h with respect to g is defined by*

$$L_g\{h(v)\}(s) = \int_b^\infty e^{-s(g(v)-g(b))}h(v)g'(v)\,dv$$

provided that the integral converges for the considered values of s.

Remark 1. *If $g(v)=v$, $b=0$; then generalized Laplace transform coincides with the classical Laplace transform.*

Definition 3. *[19] A function $l:[0,\infty)\longrightarrow\mathbb{R}$ is said to be of $g(v)$-exponential order if there exist constants $M,c,T\geq 0$ such that*

$$|l(v)| \leq Me^{cg(v)}, \quad \text{for } v\geq T.$$

It is worth noting that Definition 3 provides the functional bound of generalized exponential order, which serves as a foundational analytical tool to establish exponential bounds and guarantee asymptotic exponential stability within the generalized Laplace transform framework [21].
Jarad et al. [19] proposed the criteria for the validity of a function's generalized Laplace transform.

Lemma 1. *[19]If $l:[b,\infty)\longrightarrow\mathbb{R}$ is piecewise continuous and of $g(v)$-exponential order, then its generalized Laplace transform $L_g\{l\}(s)$ exists for $s>c$.*

The L_g of certain basic functions were specified in the subsequent lemma.

Lemma 2. *[22]Let $l,g:[b,\infty)\to\mathbb{R}$ be real-valued functions such that $g(v)$ is continuous and strictly increasing on $[b,\infty)$ Then*

(a) $L_g\left\{(g(\tau)-g(b))^\theta\right\}(s) = \dfrac{\Gamma(\theta+1)}{s^{\theta+1}}$, $Re(\theta)>0$, $s>0$.

(b) $L_g\left\{e^{c(g(\tau)-g(b))}l(\tau)\right\}(s) = L_g\{l(\tau)\}(s-c)$.

The definition of the generalized convolution integral is given as follows:

Definition 4. *[19]For two g-exponentially ordered, piecewise continuous functions ψ and φ on $[0,T]$, the generalized convolution is defined as*

$$(\psi *_g \varphi)(\upsilon) = \int_b^\upsilon \psi(\tau)\varphi\left(g^{-1}(g(\upsilon)+g(b)-g(\tau))\right)g'(\tau)\,d\tau.$$

Lemma 3. *[19]For ψ, $\varphi:[0,T]\to\mathbb{R}$ that are g-exponentially ordered and piecewise continuous,*

$$L_g\{\psi *_g \varphi\}(s) = L_g\{\psi\}(s)L_g\{\varphi\}(s).$$

Lemma 4. *[22]Let $\theta\in\mathbb{C}$ with $Re(\theta)>0$ and $\rho>0$, $m=[Re(\theta)]+1$. Then*

$$L_b\left\{\left({}_b^cD^{\theta,\rho}l\right)(v)\right\}(s) = (\rho s+1-\rho)^\theta L_g\{l(v)\}(s) - \rho\sum_{k=0}^{m-1}(\rho s+1-\rho)^{\theta-1-k}\left(D^{k,\rho}l\right)(b).$$

3. MAIN RESULTS

3.1. Problem Description

Assume ρ, $\theta\in(0,1)$ and ϕ is continuous function that strictly increases. A descriptor linear system (DLS) of the generalized Caputo proportional fractional derivative type (1) is examined:

$$(IVP)\begin{cases} F\left({}^cD^{\theta,\rho,\phi}x\right)(\vartheta) = Gx(\vartheta)+Hu(\vartheta), \\ y(\vartheta) = Cx(\vartheta)+Du(\vartheta), \end{cases} \tag{1}$$

where $x:[b,T]\to\mathbb{R}^n$, $u:[b,T]\to\mathbb{R}^m$; $F,G\in\mathbb{R}^{n\times n}$, $H\in\mathbb{R}^{n\times m}$ are any matrices. G with the property that $\det[\lambda I-G]\neq 0$.

The initial condition for (1) is given by

$$x(b) = x_b. \tag{2}$$

Now, we find the solution of (1) for two cases matrix F is non-singular and singular.

3.1.1. Solution of (1)

If matrix F is non-singular, then the solution of (1) can be find as follows:

Case 1: If $\det F\neq 0$, then by using generalized Laplace transform Definition 2, the general solution of the system (1) can be found.

Theorem 1. *For the time-invariant case, the general solution of IVP (1) if $\det F\neq 0$ is*

$$\begin{aligned} x(v) \quad &= e^{\frac{\rho-1}{\rho}(\phi(v)-\phi(b))}E_\phi\left(\rho^{-\theta}F^{-1}G(\phi(v)-\phi(b))^\theta\right)x_b \\ &+\rho^{-\theta}\int_b^v e^{\frac{\rho-1}{\rho}(\phi(v)-\phi(\tau))}(\phi(v)-\phi(\tau))^{\theta-1} \\ &\times E_{\theta,\theta}\left(\rho^{-\theta}F^{-1}G(\phi(v)-\phi(\tau))^\theta\right) \\ &\times F^{-1}Hu(\tau)\phi'(\tau)\,d\tau, \end{aligned} \tag{3}$$

where $x_b=x(b)\in\mathbb{R}^n$ represents the initial state vector at time b, $\phi(v)$ is a continuous and strictly increasing weight function, and $\rho\in(0,1]$ is the scale parameter of the generalized fractional operator. Here, $E_\theta(z)$ and $E_{\theta,\beta}(z)$ denote the standard one-parameter and two-parameter Mittag-Leffler functions, respectively, defined as:

$$E_\theta(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(j\theta+1)},$$

$$E_{\theta,\beta}(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(j\theta+\beta)}, \qquad (\theta,\beta>0,\ z\in\mathbb{C}).$$

First, we define the regularity of the system described in (1). This regularity is determined solely by the coefficient matrices F and G.

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Definition 5. *The system (1) is referred to as regular if a constant scalar $s \in \mathbb{C}$ exists for which*

$$\left| F\left(\rho s+1-\rho\right)^{\theta}-G\right| \neq 0, \tag{4}$$

for a fixed value of $\theta,\rho \in (0,1)$. In this case, we also say that the matrix pair (F,G), or the matrix pencil $F\left(\rho s+1-\rho\right)^{\theta}-G$, is regular.

Remark 2. *If $\theta=\rho=1$, then Definition 5 is the same as (see Definition 3.1., [23]).*

Example 1. *The matrix pair (F,G) is examined together with*

$$F=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, G=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let's consider $\theta=\rho=\frac{1}{2}$;

$$\det\left[F\left(\rho s+1-\rho\right)^{\theta}-G\right]=\det\begin{bmatrix} \frac{(s+1)^{\frac{1}{2}}}{\sqrt{2}} & -1 & 0 \\ 0 & \frac{(s+1)^{\frac{1}{2}}}{\sqrt{2}} & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\frac{s+1}{2}\neq 0,\ \forall s\in\mathbb{C},\ s\neq -1.$$

Thus, (F,G) is regular.

Example 2. *The matrix pair (F,G) is examined together with*

$$F=\begin{bmatrix} 0 & 0 & 0 & 1.72 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.82 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G=\begin{bmatrix} 0 & 1.1 & 0 & 0 & 0 \\ 0 & 0 & 1.56 & 0 & 0 \\ 1.23 & 0 & 0 & 1.98 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.01 & 0 & 0 \end{bmatrix}.$$

Let's assume $\theta=\rho=\frac{1}{3}$;

$$\left[F\left(\rho s+1-\rho\right)^{\theta}-G\right]$$

$$=\begin{bmatrix} 0 & -1.1 & 0 & 1.72\frac{(s+2)^{\frac{1}{3}}}{3^{\frac{1}{3}}} & 0 \\ 0 & 0 & -1.56 & 0 & 0 \\ \frac{-0.82(s+2)^{\frac{1}{3}}}{3^{\frac{1}{3}}}-1.23 & 0 & 0 & -1.98 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.01 & 0 & \frac{(s+2)^{\frac{1}{3}}}{3^{\frac{1}{3}}} \end{bmatrix}.$$

Because the fourth row of the matrix $F\left(\rho s+1-\rho\right)^{\theta}-G$ consists entirely of zeros, we obtain

$$\det\left[F\left(\rho s+1-\rho\right)^{\theta}-G\right]=0,\ \forall\, s\in\mathbb{C}.$$

Thus, (F,G) is not regular.

If matrix F is singular, then the solution of (1) can be find as follows:

Case 2: If $\det F=0$ and it is assumed that the pencil of the pair (F,G) is regular, then by using decomposition method [24], and generalized Laplace transform, Definition 2 respectively, the general solution of the system (1) can be found.

It is well-known [25, 26, p. 92] that if the pencil is regular, then there exists a pair of nonsingular matrices R, $S\in\mathbb{R}^{n\times n}$ such that

$$R\left[F\left(\rho s+1-\rho\right)-G\right]S=\begin{bmatrix} \mathbb{I}_{n_1} & O \\ O & N \end{bmatrix}\left(\rho s+1-\rho\right)-\begin{bmatrix} G_1 & O \\ O & \mathbb{I}_{n_2} \end{bmatrix}, \tag{5}$$

where n_1 is equal to the degree of the polynomial $\det\left[F\left(\rho s+1-\rho\right)-G\right]$, $G_1\in\mathbb{R}^{n_1\times n_1}$, $N\in\mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with the index μ (i.e. $N^{\mu}=0$ and $N^{\mu-1}\neq 0$) and $n_1+n_2=n$.

By the assumption (4), the pencil $\left[F\left(\rho s+1-\rho\right)^{\theta}-G\right]$ is regular and we may apply the decomposition (5) to equation (1).

$$RFS=\begin{bmatrix} \mathbb{I}_{n_1} & O \\ O & N \end{bmatrix},\ RGS=\begin{bmatrix} G_1 & O \\ O & \mathbb{I}_{n_2} \end{bmatrix}. \tag{6}$$

Premultiplying (1) by the matrix $R\in\mathbb{R}^{n\times n}$

$$RF\,{}^{c}D^{\theta,\rho,\phi}x(\vartheta)=RGx(\vartheta)+RHu(\vartheta)$$

and introducing the new state vector

$$\begin{bmatrix} x_1(\vartheta) \\ x_2(\vartheta) \end{bmatrix}=S^{-1}x(\vartheta),$$

$$RFS\,{}^{c}D^{\theta,\rho,\phi}S^{-1}x(\vartheta)=RGSS^{-1}x(\vartheta)+RHu(\vartheta)$$

where $S\in\mathbb{R}^{n\times n}$, $x_1(\vartheta)\in\mathbb{R}^{n_1}$ and $x_2(\vartheta)\in\mathbb{R}^{n_2}$, we obtain

$$\begin{bmatrix} \mathbb{I}_{n_1} & O \\ O & N \end{bmatrix}{}^{c}D^{\theta,\rho,\phi}\begin{bmatrix} x_1(\vartheta) \\ x_2(\vartheta) \end{bmatrix}=\begin{bmatrix} G_1 & O \\ O & \mathbb{I}_{n_2} \end{bmatrix}\begin{bmatrix} x_1(\vartheta) \\ x_2(\vartheta) \end{bmatrix}+RHu(\vartheta).$$

Therefore, we have

$$\mathbb{I}_{n_1}\,{}^{c}D^{\theta,\rho,\phi}x_1(\vartheta)=G_1x_1(\vartheta)+H_1u(\vartheta),\ \text{(Slow Sub-system)} \tag{7}$$

$$N\,{}^{c}D^{\theta,\rho,\phi}x_2(\vartheta)=\mathbb{I}_{n_2}x_2(\vartheta)+H_2u(\vartheta),\ \text{(Fast Sub-system)}, \tag{8}$$

where

$$RH=\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, H_1\in\mathbb{R}^{n_1\times m}, H_2\in\mathbb{R}^{n_2\times m}.$$

Using Theorem 1, the general solution to the equation (7) can be found as follows:

$$\begin{aligned} x_1(\vartheta) &= e^{\frac{\rho-1}{\rho}(\phi(\vartheta)-\phi(b))}E_{\theta}\left(\rho^{-\theta}G_1\left(\phi(\vartheta)-\phi(b)\right)^{\theta}\right)x_{1b} \\ &+\rho^{-\theta}\int_b^{\vartheta}e^{\frac{\rho-1}{\rho}(\phi(\vartheta)-\phi(\tau))}\left(\phi(\vartheta)-\phi(\tau)\right)^{\theta-1} \\ &\times E_{\theta,\theta}\left(\rho^{-\theta}G_1\left(\phi(\vartheta)-\phi(\tau)\right)^{\theta}\right)H_1u(\tau)\phi'(\tau)\,d\tau, \end{aligned} \tag{9}$$

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where

$$E_\theta\left(\rho^{-\theta}G_1\left(\phi\left(\vartheta\right)-\phi\left(b\right)\right)^\theta\right)=\sum_{j=0}^{\infty}\frac{\left[\rho^{-\theta}G_1(\phi(\vartheta)-\phi(b))^\theta\right]^j}{\Gamma(j\theta+1)},$$
$$E_{\theta,\theta}\left(\rho^{-\theta}G_1\left(\phi\left(\vartheta\right)-\phi\left(\tau\right)\right)^\theta\right)=\sum_{j=0}^{\infty}\frac{\left[\rho^{-\theta}G_1(\phi(\vartheta)-\phi(\tau))^\theta\right]^j}{\Gamma[\theta(j+1)]}$$

and $x_{1b}\in\mathbb{R}^{n_1}$ is the initial condition for (7) defined by

$$\begin{bmatrix}x_{1b}\\x_{2b}\end{bmatrix}=S^{-1}x_b,\ x_b=x\left(b\right).$$

To find the solution of equation (8), we apply the generalized Laplace transform and obtain

$$\begin{aligned}&N(\rho s+1-\rho)^\theta X_2(s)-N\rho(\rho s+1-\rho)^{\theta-1}x_{2b}\\&=\mathbb{I}_{n_2}X_2(s)+H_2U(s).\end{aligned}\tag{10}$$

Since by Lemma 4, for $0<\theta\leq 1$,

$$\begin{aligned}L_\phi\left[{}^cD^{\theta,\rho,\phi}x_2(\vartheta)\right]&=(\rho s+1-\rho)^\theta X_2(s)\\&\quad-\rho(\rho s+1-\rho)^{\theta-1}x_{2b},\end{aligned}$$

where $X_2\left(s\right)=L_\phi\left[x_2\left(\vartheta\right)\right]$.

From (10) we have

$$\begin{aligned}X_2(s)=&\left[N(\rho s+1-\rho)^\theta-\mathbb{I}_{n_2}\right]^{-1}\\&\left[H_2U(s)+N\rho(\rho s+1-\rho)^{\theta-1}x_{2b}\right].\end{aligned}\tag{11}$$

It is easy to check that

$$\left[N\left(\rho s+1-\rho\right)^\theta-\mathbb{I}_{n_2}\right]^{-1}=-\sum_{i=0}^{\mu-1}N^i\left(\rho s+1-\rho\right)^{\theta i},\tag{12}$$

since

$$\left[N\left(\rho s+1-\rho\right)^\theta-\mathbb{I}_{n_2}\right]\left(-\sum_{i=0}^{\mu-1}N^i\left(\rho s+1-\rho\right)^{\theta i}\right)=\mathbb{I}_{n_2}$$

and $N^i=0$ for $i=\mu,\mu+1,\cdots$.

Substitution of (12) into (11) yields

$$\begin{aligned}X_2(s)=&-H_2U(s)-\rho Nx_{2b}\frac{1}{\left(s-\frac{\rho-1}{\rho}\right)^{1-\theta}}\\&-\sum_{i=1}^{\mu-1}\left[N^iH_2(\rho s+1-\rho)^{\theta i}U(s)\right.\\&\left.+N^{i+1}\rho(\rho s+1-\rho)^{(i+1)\theta-1}x_{2b}\right].\end{aligned}\tag{13}$$

Applying the inverse generalized Laplace transform (L_ϕ^{-1}) to (13) and then the Lemma 3, we obtain

$$\begin{aligned}&x_2\left(\vartheta\right)\\&=-H_2u\left(\vartheta\right)-\rho Nx_{2b}\frac{e^{\left(\frac{\rho-1}{\rho}\right)(\phi(\vartheta)-\phi(b))}\left(\phi\left(\vartheta\right)-\phi\left(b\right)\right)^{-\theta}}{\Gamma\left(1-\theta\right)}\\&-\sum_{i=1}^{\mu-1}\left[N^iH_2\ {}^cD^{\theta i,\rho,\phi}u\left(\vartheta\right)+\rho N^{i+1}\ {}^cD^{(i+1)\theta-1,\rho,\phi}x_{2b}\right],\end{aligned}\tag{14}$$

since,

$$L_\phi^{-1}\left[\frac{1}{\left(s-\frac{\rho-1}{\rho}\right)^{\theta+1}}\right](\vartheta)=\frac{e^{\left(\frac{\rho-1}{\rho}\right)(\phi(\vartheta)-\phi(b))}\left(\phi\left(\vartheta\right)-\phi\left(b\right)\right)^\theta}{\Gamma\left(\theta+1\right)},$$

and

$$\begin{aligned}&L_\phi^{-1}\left[(\rho s+1-\rho)^\theta X_2(s)-\rho(\rho s+1-\rho)^{\theta-1}x_{3b}\right](\vartheta)\\&={}^cD^{\theta,\rho,\phi}x_2(\vartheta).\end{aligned}$$

Therefore, the following theorem has been proved.

Theorem 2. *The solution to the state equation (1) with the initial conditions (2) has the form*

$$x\left(\vartheta\right)=S\begin{bmatrix}x_1\left(\vartheta\right)\\x_2\left(\vartheta\right)\end{bmatrix},$$

$$\begin{aligned}x\left(\vartheta\right)=&\left(RG\right)^{-1}\begin{bmatrix}G_1&O\\O&\mathbb{I}_{n_2}\end{bmatrix}\\&\times\begin{bmatrix}e^{\frac{\rho-1}{\rho}(\phi(\vartheta)-\phi(b))}E_\theta\left(\rho^{-\theta}G_1\left(\phi\left(\vartheta\right)-\phi\left(b\right)\right)^\theta\right)x_{1b}\\+\rho^{-\theta}\int_b^\vartheta e^{\frac{\rho-1}{\rho}(\phi(\vartheta)-\phi(\tau))}\left(\phi\left(\vartheta\right)-\phi\left(\tau\right)\right)^{\theta-1}\\\times E_{\theta,\theta}\left(\rho^{-\theta}G_1\left(\phi\left(\vartheta\right)-\phi\left(\tau\right)\right)^\theta\right)H_1u\left(\tau\right)\phi'\left(\tau\right)d\tau\\-H_2u\left(\vartheta\right)-\rho Nx_{2b}\frac{e^{\left(\frac{\rho-1}{\rho}\right)(\phi(\vartheta)-\phi(b))}\left(\phi\left(\vartheta\right)-\phi\left(b\right)\right)^{-\theta}}{\Gamma\left(1-\theta\right)}\\-\sum_{i=1}^{\mu-1}\left[N^iB_2\ {}^cD^{\theta i,\rho,\phi}u\left(\vartheta\right)+\rho N^{i+1}\ {}^cD^{(i+1)\theta-1,\rho,\phi}x_{2b}\right]\end{bmatrix},\end{aligned}\tag{15}$$

where $x_1\left(\vartheta\right)$ and $x_2\left(\vartheta\right)$ are given by (9) and (14), respectively.

Knowing the solution (15), we can find the output $y\left(\vartheta\right)$ of the system using the formula

$$y\left(\vartheta\right)=CS\begin{bmatrix}x_1\left(\vartheta\right)\\x_2\left(\vartheta\right)\end{bmatrix}+Du\left(\vartheta\right).\tag{16}$$

Example 3. *The following electrical circuit, adapted from [24], is considered to illustrate the applicability of our proposed generalized fractional proportional descriptor system.*

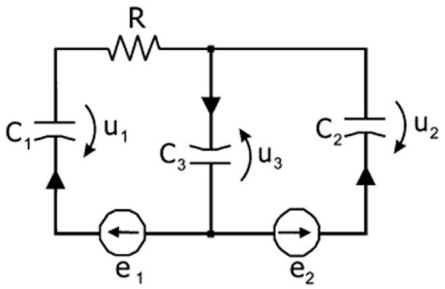


Fig. 1. Descriptor electrical circuit adapted from [24]

.

Using the Kirchhoff's laws we can write the following equa-

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tions for the electrical circuit:

$$e_1 = RC_1\,{}^cD^{\theta,\rho,\phi}u_1(\vartheta) + u_1 + u_3,$$

$$C_1\,{}^cD^{\theta,\rho,\phi}u_1(\vartheta) + C_2\,{}^cD^{\theta,\rho,\phi}u_2(\vartheta) - C_3\,{}^cD^{\theta,\rho,\phi}u_3(\vartheta) = 0,$$

$$e_2 = u_2 + u_3.$$

The equations can be written in the form

$$\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix} {}^cD^{\theta,\rho,\phi} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

In this case we have

$$F = \begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the matrix F is singular ($\det F = 0$) but

$$\det\left[F(\rho s + 1 - \rho)^\theta - G\right]$$

$$= \begin{vmatrix} RC_1(\rho s+1-\rho)^\theta + 1 & 0 & (\rho s+1-\rho)^\theta \\ C_1(\rho s+1-\rho)^\theta & C_2(\rho s+1-\rho)^\theta & -C_3(\rho s+1-\rho)^\theta \\ 0 & 0 & (\rho s+1-\rho)^\theta + 1 \end{vmatrix}$$

$$= RC_1C_2(\rho s+1-\rho)^{3\theta} + (RC_1(C_2+C_3) + C_1 + C_2)(\rho s+1-\rho)^{2\theta} + (C_2+C_3)(\rho s+1-\rho)^\theta.$$

Therefore,

$$\det\left[F(\rho s+1-\rho)^\theta - G\right] \neq 0,$$

for admissible values of the circuit parameters. Hence, the matrix pencil $F(\rho s+1-\rho)^\theta - G$ is regular and condition (4) is satisfied. Consequently, the considered electrical circuit represents a regular descriptor fractional proportional linear system. It is worth emphasizing that the purpose of this example is to illustrate the applicability of the theoretical results established in this paper. In particular, the descriptor model is analyzed within the generalized fractional proportional framework developed herein, which is different from the formulation considered in [24].

3.2. Stability Analysis

Stability is a well-recognized essential property of any practical system. Without stability, the system may malfunction or even become damaged during use. Therefore, ensuring stability is of utmost importance. In control system design, our primary focus is on the stability of closed-loop systems. In this section and the next, we will explore the stability of regular descriptor linear system (RDLS).

3.2.1. Stability of Generalized Caputo Fractional Systems

To investigate the dynamic behavior and state trajectories of the unconstrained descriptor system, we consider the following autonomous homogeneous state equation:

$$\begin{cases} \left({}^cD^{\theta,\rho,\phi}x\right)(\nu) = Gx(\nu), \\ x(b) = x_b. \end{cases} \tag{17}$$

Definition 6. *Let ϕ is continuous, strictly increasing function. The trivial solution of system (17) or*

$$F\left({}^cD^{\theta,\rho,\phi}x\right)(\vartheta) = Gx(\vartheta),\ x(b) = x_b),$$

is called Mittag-Leffler stable if

$$\|x(\vartheta)\| \leq \left[x(\vartheta_0)E_\theta\left(-\lambda\rho^{-\theta}(\phi(\vartheta) - \phi(\vartheta_0))^\theta\right)\right]^\gamma, \quad \text{for all } \vartheta > \vartheta_0 \tag{18}$$

where ϑ_0 is the initial time, $\lambda \geq 0$, for fixed ρ, $\theta \in (0,1)$ and $\gamma > 0$.

Definition 7. *The system (17) or*

$$F\left({}^cD^{\theta,\rho,\phi}x\right)(\vartheta) = Gx(\vartheta),\ x(b) = x_b),$$

is called asymptotically stable if it is Mittag-Leffler stable and $\lim_{\vartheta\to\infty} x(\vartheta) = 0$.

Remark 3. *Mittag-Leffler stability generalizes exponential stability and directly implies asymptotic stability. Notably, as the fractional order $\theta \to 1.0$, Mittag-Leffler stability reduces directly to classical exponential stability.*

Theorem 3. *If all eigenvalues of G satisfy*

$$|\arg(\lambda(G))| > \frac{\theta\pi}{2} \tag{19}$$

then the zero solution of system (17) is asymptotically stable.

Proof. [20, Theorem 2.3] immediately follows that the solution of system (17) is given by

$$x(v) = e^{\frac{\rho-1}{\rho}(\phi(v)-\phi(b))} E_\theta\left(\rho^{-\theta}G(\phi(v) - \phi(b))^\theta\right)x_b.$$

First, suppose the matrix G is diagonalizable, i.e. there exists an invertible matrix T such that

$$\Lambda = T^{-1}GT = diag(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Then,

$$\begin{aligned} &E_\theta\left(\rho^{-\theta}G(\phi(v) - \phi(b))^\theta\right) \\ &= T\,E_\theta\left(\rho^{-\theta}G(\phi(v) - \phi(b))^\theta\right)T^{-1} \\ &= T\ \text{diag}\left(E_\theta\left(\rho^{-\theta}\lambda_1(\phi(v) - \phi(b))^\theta\right),\right. \\ &\left. E_\theta\left(\rho^{-\theta}\lambda_2(\phi(v) - \phi(b))^\theta\right), \quad \ldots, \right. \\ &\left. E_\theta\left(\rho^{-\theta}\lambda_n(\phi(v) - \phi(b))^\theta\right)\right)T^{-1}. \end{aligned}$$

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Applying (19) and [7, equation (2.8′)], we have

$$\begin{aligned}
&E_\theta\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)\\
&=-\sum_{k=1}^{p}\frac{\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)^{-k}}{\Gamma(1-k\theta)}\\
&\quad+O\left(\left|\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right|\right)\\
&\longrightarrow 0, \qquad \text{as } v\to\infty,\ 1\le i\le n.
\end{aligned}$$

Thus

$$\left\|E_\theta\left(H\vartheta^\theta\right)\right\| = \left\|\operatorname{diag}\begin{pmatrix} E_\theta\left(\rho^{-\theta}\lambda_1\left(\phi(v)-\phi(b)\right)^\theta\right),\\ E_\theta\left(\rho^{-\theta}\lambda_2\left(\phi(v)-\phi(b)\right)^\theta\right),\\ \vdots\\ E_\theta\left(\rho^{-\theta}\lambda_n\left(\phi(v)-\phi(b)\right)^\theta\right)\end{pmatrix}\right\| \longrightarrow 0.$$

Hence the conclusion holds.

Next, suppose the matrix G is similar to a Jordan canonical form, i.e; there exists an invertible matrix T such that

$$J = T^{-1}GT = diag\left(J_1, J_2, \ldots, J_r\right),$$

where J_i, $1\le i\le r$ has the following form

$$\begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1\\ & & & \lambda_i\end{pmatrix}_{n_i\times n_i},$$

and $\sum_{i=1}^{r} n_i = n$. Obviously,

$$\begin{aligned}
&E_\theta\left(\rho^{-\theta}G\left(\phi(v)-\phi(b)\right)^\theta\right)\\
&= T\ \operatorname{diag}\begin{pmatrix} E_\theta\left(\rho^{-\theta}J_1\left(\phi(v)-\phi(b)\right)^\theta\right),\\ E_\theta\left(\rho^{-\theta}J_2\left(\phi(v)-\phi(b)\right)^\theta\right),\\ \vdots\\ E_\theta\left(\rho^{-\theta}J_r\left(\phi(v)-\phi(b)\right)^\theta\right)\end{pmatrix} T^{-1}.
\end{aligned}$$

where $1\le i\le r$,

$$\begin{aligned}
&E_\theta\left(\rho^{-\theta}J_i\left(\phi(v)-\phi(b)\right)^\theta\right)\\
&=\sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}J_i\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)}\\
&=\sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\left(\phi(v)-\phi(b)\right)^\theta\right)^k J_i^k}{\Gamma(\theta k+\theta)}\\
&=\sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)}\\
&\quad\begin{pmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \cdots & \binom{k}{n_i-1}\lambda_i^{k-n_i+1}\\ 0 & \lambda_i^k & \ddots & \vdots\\ \vdots & 0 & \ddots & \binom{k}{1}\lambda_i^{k-1}\\ 0 & \cdots & 0 & \lambda_i^k\end{pmatrix}.
\end{aligned}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)} & \sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)}C_k^1\lambda_i^{k-1} & \cdots & \sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)}C_k^{n_i-1}\lambda_i^{k-n_i+1}\\ & \sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)}\lambda_i^k & \ddots & \vdots\\ & & \ddots & \sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)}C_k^1\lambda_i^{k-1}\\ & & & \sum_{k=0}^{\infty}\frac{\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)^k}{\Gamma(\theta k+\theta)}\end{pmatrix},$$

where C_k^j, $1\le j\le n_i-1$ are the binomial coefficients

$$\begin{pmatrix} E_\theta(\xi_i) & \frac{1}{1!}\frac{\partial}{\partial\lambda_i}E_\theta(\xi_i) & \cdots & \frac{1}{(n_i-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{n_i-1}E_\theta(\xi_i)\\ 0 & E_\theta(\xi_i) & \ddots & \vdots\\ \vdots & \ddots & \ddots & \frac{1}{1!}\frac{\partial}{\partial\lambda_i}E_\theta(\xi_i)\\ 0 & \cdots & 0 & E_\theta(\xi_i)\end{pmatrix},$$

where

$$\xi_i = \rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta.$$

By some tedious calculations and [7, Remark 2.1], if $\left|\arg\left(\lambda_i(G)\right)\right| > \frac{\theta\pi}{2}$, $1\le i\le r$ and $v\to\infty$, then we have

$$\begin{aligned}
&\left|E_\theta\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)\right| &&\longrightarrow 0,\\
&\left|\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j E_\theta\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)\right| &&\longrightarrow 0,\\
&0\le j\le n_i-1,\ 1\le i\le r.
\end{aligned}$$

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Indeed, these can be seen from the following:

$$\begin{aligned}
&E_\theta\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)\\
&=-\sum_{k=2}^{p}\frac{\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)^{-k}}{\Gamma(\theta-\theta k)}\\
&\quad+O\left(\left|\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right|^{-1-p}\right).
\end{aligned}$$

This implies that

$$\left|E_\theta\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)\right|\to 0 \text{ as } v\to\infty,$$

and

$$\begin{aligned}
&\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j E_\theta\left(\rho^{-\theta}\lambda_i(\phi(v)-\phi(b))^\theta\right)\\
&=\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j\left\{-\sum_{k=2}^{p}\frac{\left(\rho^{-\theta}\lambda_i(\phi(v)-\phi(b))^\theta\right)^{-k}}{\Gamma(\theta-\theta k)}\right.\\
&\qquad\left.+O\left(\left|\rho^{-\theta}\lambda_i(\phi(v)-\phi(b))^\theta\right|^{-1-p}\right)\right\}\\
&=-\sum_{k=2}^{p}\frac{(-1)^j(k+j-1)\cdots(k+1)k\left(\rho^{-\theta}\lambda_i\right)^{-k-j}(\phi(v)-\phi(b))^{-\theta k}}{j!\Gamma(\theta-\theta k)}\\
&\quad+O\left(\left|\rho^{-\theta}\lambda_i\right|^{-1-p-j}\left|(\phi(v)-\phi(b))^\theta\right|^{-1-p}\right)\\
&=-\sum_{k=2}^{p}\frac{(-1)^j(k+j-1)!\left(\rho^{-\theta}\lambda_i\right)^{-k-j}(\phi(v)-\phi(b))^{-\theta k}}{j!(k-1)!\Gamma(\theta-\theta k)}\\
&\quad+O\left(\left|\rho^{-\theta}\lambda_i\right|^{-1-p-j}\left|(\phi(v)-\phi(b))^\theta\right|^{-1-p}\right).
\end{aligned}$$

Which leads to $\left|\frac{1}{j!}\left(\frac{\partial}{\partial\lambda_i}\right)^j E_\theta\left(\rho^{-\theta}\lambda_i\left(\phi(v)-\phi(b)\right)^\theta\right)\right|\to 0$, $1\le j\le n_i-1$ as $v\to\infty$. It now follows that

$$\|x(v)\|=\left\|e^{\frac{\rho-1}{\rho}(\phi(v)-\phi(b))}E_\theta\left(\rho^{-\theta}G\left(\phi(v)-\phi(b)\right)^\theta\right)x_b\right\|\to 0 \quad \text{as } v\to\infty,$$

for any non-zero initial value x_b. The proof is complete. □

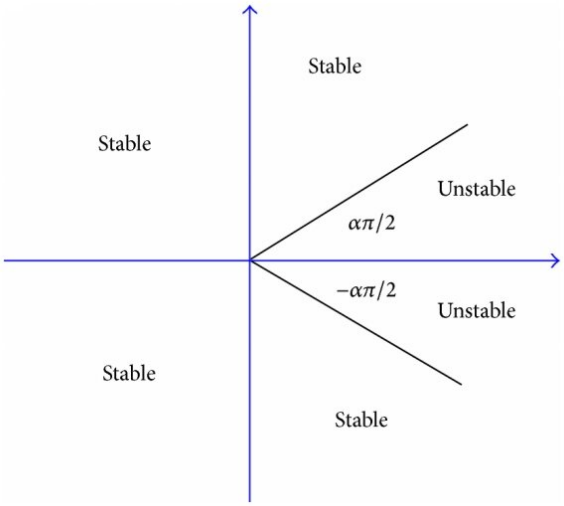


Fig. 2. Stability region of the generalized fractional linear control system based on Matignon's criteria [8]

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3.2.2. Stabilty of Descriptor Generalized Caputo Fractional Systems Similar to the case of normal linear systems, when examining the stability of DLS, we need to focus solely on the following generalized Caputo fractional proportional linear homogeneous equation

$$F\left({}^cD^{\theta,\rho,\phi}x\right)(\vartheta)=Gx(\vartheta),\ x(b)=x_b, \tag{20}$$

where $x\in\mathbb{R}^n$ is the state vector, and $F,G\in\mathbb{R}^{n\times n}$ are the coefficient matrices. To simplify our discussion, we assume that the system (20) is regular, as this guarantees that the system has a unique solution.

Definition 8. *For a fixed θ, $\rho\in(0,1)$. Let's establish the set of finite poles of the system as:*

$$\begin{aligned}
&\sigma_{\theta,\rho}(F,G)\\
&=\left\{s: s\in\mathbb{C},\ s \text{ finite},\ \det\left[F(\rho s+1-\rho)^\theta-G\right]=0\right\}.
\end{aligned}$$

It has a maximum of $n_o=\text{rank}F\le n$ complex numbers.

$\sigma_{\theta,\rho}(F,G)$ generalizes related to different values of θ,ρ.

Remark 4. *$\sigma_{1,1}(F,G)=\sigma(F,G)$ which is same as (see, equation (3.6.1), [23]).*

Example 4. *Consider the matrix pair (F,G) with*

$$F=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\ G=\begin{bmatrix}-1&0&2&1\\0&-2&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

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Let's consider $\theta=\rho=\frac{1}{2}$;

$$\det\left[F(\rho s+1-\rho)^{\theta}-G\right]$$
$$=\det\begin{bmatrix}\sqrt{\frac{1}{2}(s+1)}+1 & 0 & -2 & -1\\ 0 & 2 & 0 & 0\\ 0 & 0 & -1 & \sqrt{\frac{1}{2}(s+1)}\\ 0 & 0 & \sqrt{\frac{1}{2}(s+1)} & -1\end{bmatrix}$$
$$=\left(\sqrt{\frac{1}{2}(s+1)}+1\right)(1-s),\quad \forall s\in\mathbb{C}.$$

Thus,

$$\sigma_{\frac{1}{2},\frac{1}{2}}(F,G)=\left\{1,\ 2i^4-1\right\}.$$

Definition 9. *For fixed* $\theta,\ \rho\in(0,1)$, *let us define the stable set:*

$$\mathbb{C}^{-}_{\theta,\rho}=\left\{s:s\in\mathbb{C},Re\left[(\rho s+1-\rho)^{\theta}\right]<0\right\}.$$

$\mathbb{C}^{-}_{\theta,\rho}$ generalizes related to different values of θ,ρ.

Remark 5. $\mathbb{C}^{-}_{1,1}=\mathbb{C}^{-}$ *which is same as [23].*

Example 5. *Let's consider* $\theta=0.3,\ \rho=0.8$; *then*

$$\mathbb{C}^{-}_{0.3,0.8}=\left\{s:s\in\mathbb{C},Re\left[(0.8s+0.2)^{0.3}\right]<0\right\},$$

which can be evaluated numerically or graphically to find specific regions in the complex plane where this condition is satisfied. Let

$$z=0.8s+0.2.$$

Write z in polar form,

$$z^{0.3}=r^{0.3}e^{i(0.3\theta)}.$$

The real part of $z^{0.3}$ *is:*

$$Re\left[z^{0.3}\right]=r^{0.3}\cos(0.3\theta).$$

We want $r^{0.3}\cos(0.3\theta)<0$. *Since* $r^{0.3}\geq 0$, *for the inequality*

$$r^{0.3}\cos(0.3\theta)<0,$$

to hold, we must have: $\cos(0.3\theta)<0$. *Determine the range for* θ : *The cosine function is negative in the intervals:*

$$\tfrac{\pi}{2}+2k\pi<0.3\theta<\tfrac{3\pi}{2}+2k\pi,\ k\in\mathbb{Z}.$$

To find the corresponding intervals for θ, *we solve the inequality for* θ. *For* $k=0$:

$$\frac{\frac{\pi}{2}}{0.3}<\theta<\frac{\frac{3\pi}{2}}{0.3},$$
$$\tfrac{5\pi}{3}<\theta<5\pi,$$

thus, $\theta\in\left(\frac{5\pi}{3},5\pi\right)$. *Substitute back to find s, since* $z=0.8s+0.2$, *the argument* θ *of z will depend on s.*

$\sigma_{\theta,\rho}(I,G)$ is frequently written as $\sigma_{\theta,\rho}(G)$ for simplicity.

Theorem 4. *Let the DLS (20) is regular, and assume R and S are two non-singular matrices satisfying (6). Then, the following relation holds*

$$\sigma_{\theta,\rho}(F,G)=\sigma_{\theta,\rho}(G_1).$$

Proof. Given that N is a nilpotent matrix, it follows that:

$$\det\left[N(\rho s+1-\rho)^{\theta}-I_{n_2}\right]=(-1)^{n_2},$$

therefore,

$$\sigma_{\theta,\rho}(N,I_{n_2})=\phi.$$

By applying this relation along with those in (6), we obtain:

$$\begin{aligned}&\sigma_{\theta,\rho}(F,G)=\sigma_{\theta,\rho}(SFR,SGR)\\&=\sigma_{\theta,\rho}\left(\begin{bmatrix}\mathbb{I}_{n_1} & O\\ O & N\end{bmatrix},\begin{bmatrix}G_1 & O\\ O & \mathbb{I}_{n_2}\end{bmatrix}\right)\\&=\sigma_{\theta,\rho}(I,G_1)\cup\sigma_{\theta,\rho}(N,I_{n_2})=\sigma_{\theta,\rho}(I,G_1)\cup\phi\\&=\sigma_{\theta,\rho}(G_1).\end{aligned}$$

The proof is then completed. □

Remark 6. *If* $\theta=\rho=1$, *then Theorem 4 is same as (see, Lemma 3.1., [23]).*

The following theorem establishes a definitive criterion for the stability of RDLS. It clearly indicates that the stability of the system (20) is determined by $\sigma_{\theta,\rho}(F,G)$.

Theorem 5. *The RDLS (20) is stable if and only if*

$$\sigma_{\theta,\rho}(F,G)\ \textit{lies entirely within}\ \mathbb{C}^{-}_{\theta,\rho}. \tag{21}$$

Proof. The proof employs the standard decomposition of RDLS. As stated in [23, Theorem 3.5], two nonsingular matrices S and R can be found such that (6) is satisfied. Consequently, by setting

$$x(\vartheta)=R\begin{bmatrix}x_1(\vartheta)\\ x_2(\vartheta)\end{bmatrix},\ x_b=R\begin{bmatrix}x_{1b}\\ x_{2b}\end{bmatrix}. \tag{22}$$

The system (20) is equivalently transformed into the following systems

$${}^{c}D^{\theta,\rho,\phi}x_1(\vartheta)=G_1x_1(\vartheta),\ x_1(b)=x_{1b}, \tag{23}$$

and

$$N{}^{c}D^{\theta,\rho,\phi}x_2(\vartheta)=x_2(\vartheta),\ x_2(b)=x_{2b}, \tag{24}$$

where N is a nilpotent matrix.
[20, Theorem 2.3] implies that the system (23) possesses a solution

$$\begin{aligned}&x_1(\vartheta)\\&=e^{\frac{\rho-1}{\rho}(\phi(\vartheta)-\phi(0))}E_{\theta}\left(\rho^{-\theta}G(\phi(\vartheta)-\phi(0))^{\theta}\right)x_{1b},\quad \vartheta>b,\end{aligned}$$

and it follows from Theorem 2 that the system (24) has the solution

$$x_2(\vartheta)=0,\ \vartheta>b. \tag{25}$$

Now let inequality (18) hold. Then we have, using (22) and (25)

$$\begin{aligned}&\|x_1(\vartheta)\|\\&=\left\|R^{-1}x(\vartheta)\right\|=\left\|R^{-1}\right\|\|x(\vartheta)\|\\&\leq\left\|R^{-1}\right\|\left\|\left[x(b)E_{\theta}\left(-\lambda\rho^{-\theta}(\phi(\vartheta)-\phi(b))^{\theta}\right)\right]\right\|\\&\leq\left\|R^{-1}\right\|E_{\theta}\left(-\lambda\rho^{-\theta}(\phi(\vartheta)-\phi(b))^{\theta}\right)\|x(b)\|,\ \vartheta>b.\end{aligned}$$

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Consequently, system (23) is asymptotically stable, so

$$\sigma_{\theta,\rho}(G_1) \subset \mathbb{C}^{-}_{\theta,\rho}. \tag{26}$$

Conversely, if (26) holds, positive scalars λ and θ can be determined such that

$$\|x_1(\vartheta)\| \leq \left\|R^{-1}\right\| E_\theta\left(-\lambda\left(\phi(\vartheta)-\phi(b)\right)^\theta\right)\|x(b)\|, \ \vartheta > b,$$

with this, (22) and (25), we can easily obtain

$$\begin{aligned}\|x(\vartheta)\| &\leq \|R\|\left\|\begin{bmatrix} x_1(\vartheta) \\ x_2(\vartheta)\end{bmatrix}\right\| \\ &= \|R\|\,\|x_1(\vartheta)\| \\ &\leq \|R\|\left\|R^{-1}\right\| E_\theta\left(-\lambda\rho^{-\theta}\left(\phi(\vartheta)-\phi(b)\right)^\theta\right)\|x(b)\| \\ &\leq E_\theta\left(-\lambda\rho^{-\theta}\left(\phi(\vartheta)-\phi(b)\right)^\theta\right)\|x(b)\|, \ \vartheta > b.\end{aligned}$$

This is identical to (18). From the reasoning provided earlier, we can conclude that the system is stable if and only if (26) is satisfied. Additionally, based on Theorem 4, we know that the system is stable if and only if (21) holds true. Therefore, the second conclusion of the theorem is valid. □

Remark 7. *If $\theta = \rho = 1$, then Theorem 5 is same as (see, Theorem 3.14., [23]).*

4. CONCLUDING REMARKS AND FUTURE DIRECTIONS

This study examined the stability of regular descriptor fractional proportional linear control systems governed by generalized Caputo derivatives. Classical notions of stability were extended to fractional-order systems by introducing Mittag–Leffler stability, which generalizes exponential stability, along with asymptotic stability criteria. Theoretical developments, supported by illustrative examples, demonstrated that stability is fundamentally influenced by the eigenvalues of the system matrix G and the generalized fractional orders θ and ρ. It was established that the zero solution of a fractional linear control system is asymptotically stable when the eigenvalues of G satisfy specific angular constraints. For descriptor systems, stability was further characterized through the analysis of the pole set $\sigma_{\theta,\rho}(F,G)$, requiring its elements to lie within the stable region $\mathbb{C}^{-}_{\theta,\rho}$ in the complex plane. The presented results provide a rigorous framework for stability assessment and significantly extend the foundations of linear system theory to fractional-order settings, with potential applications in future studies on controllability, observability, and optimal control of systems with regular pencils. The established analytical criteria provide useful mathematical tools for physical system modeling (such as constrained electrical networks and aerodynamic systems) and the synthesis of robust feedback controllers. Future work will extend these stability results to dynamic feedback control, robust stabilization under parameter uncertainties, and optimal control of fractional descriptor systems with regular pencils.

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