

ACCURACY COMPARISON OF TIME DELAY ESTIMATION OF RANDOM SIGNALS USING THE SINGLE-POINT CROSS-SPECTRUM PHASE AND DIRECT CROSS-CORRELATION

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Abstract

Time delay estimation (TDE) is a common issue in signal processing. For random signals, various statistical methods of analysis are used. In this work, standard uncertainties (standard deviations) of the TDE were analytically analysed and compared for two methods: direct cross-correlation function (CCF) and phase of the cross-spectral density function (CSDF). In the frequency domain, the CSDF-phase single-point method was considered, consisting of determining the time delay for one selected frequency. The metrological properties of the CCF and the single-point phase method are presented for three models of stochastic signals contaminated with white noise interferences. For these models, the standard uncertainties of the TDE have been compared taking advantage phase CSDF and CCF for the signal-to-noise ratio (SNR) in the range $0.1 \leq \text{SNR} \leq 100$. It was found that for small SNR values ($\text{SNR} < 1$) the CCF method is more accurate. On the other hand, for larger SNR values (especially above one), lower values of the standard uncertainty of the time delay are obtained for the phase method. Specific SNR values for which the single-point phase method is more accurate depend on the adopted models and parameters of the signals and on the parameters of the analysis.

Keywords: time delay estimation, stochastic signals, cross-spectral density, cross-correlation, standard uncertainty.

1. Introduction

Applications of statistical methods for *time delay estimation* (TDE) of measurement random signals, received from spatially separated sensors, are widely described in the literature. In recent decades, several TDE methods have been elaborated based on the analysis of signals in the time, frequency and state-space domains [1–9]. These methods find application, *e.g.* in radar technology, telecommunications, acoustics, mechanics, seismology, and medicine; they aid in detecting pipeline leaks and assessing two-phase fluid flow [10–17].

The practical application of individual methods of analysis relies on the characteristics of the signals being measured. Their properties such as: stationarity, *probability distribution function*

(PDF), correlation, and SNR are important. The best-known time domain TDE method for Gaussian stationary signals is the *cross-correlation function* (CCF) [18–20]. In the classic cross-correlation method, the estimation of the time delay τ_0 relies upon the location of the position of the main CCF peak. Within the frequency domain, one can use a *cross-spectral density function* (CSDF) for the purpose of TDE [19, 21–23]. To determine the delay for the above-mentioned signals, the following methods can also be applied: differential and combined functions [16, 24, 25], conditional averaging [26, 27], and cross-correlation with the Hilbert Transform [28, 29].

The phase method, described in the 1980s [21, 30], is relatively less popular because its practical implementation is more complicated than in the case of CCF. Efficient execution of this approach necessitates obtaining an unwrapped phase spectrum estimate and extracting the time delay, which can be achieved through different approaches. The phase method can be implemented as single-point or multipoint [19, 21, 31]. The single-point method involves determining the time delay for a selected frequency f_0 and is useful when we are interested in the value of τ_0 for that particular frequency. Since the CSDF phase for nondispersive media is a linear function of the frequency in the band of the analysed signals, the multipoint method can then be used. It consists of approximating a selected range of the CSDF phase with a straight line and calculating the delay as the slope of this line. Examples of the application of the phase method in science and technology are described in the literature: this method is used in electrical capacitance tomography [22], two-phase flow measurements [32], to analyse vibration and acoustic signals [15, 17, 33–37], as well as to signals in medical and biological investigations [14, 38, 39]. It is generally known that the cross-correlation method is suitable for estimating the delay time of signals in the presence of interference, especially for unfavourable SNR. However, there are relatively few works devoted to analysing the properties of the phase method, especially the single-point phase method.

In this paper, an analysis and comparison of the standard uncertainties for the single-point phase CSDF and the direct CCF estimator, considering three models of mutually time-delayed and noise-affected stochastic signals are presented. Since no such comparison was found in the available literature, the author believes that this work fills this research gap. Section 2 presents the analysed models of measurement signals. It was assumed that the analysed signals are realizations of a stationary and ergodic random process. Section 3 briefly describes the principle of the correlation method and gives the TDE standard uncertainties based on the CCF for the adopted signal models. In Section 4, an analogous examination for the single-point phase method is presented. Section 5 provides a comparison of the standard uncertainties of TDE by CCF and the single-point phase method. Section 6 delivers the conclusions from the conducted research.

This paper is significantly improved and expanded version of conference work [40].

2. Models of measuring signals

The relationship between two mutually time-delayed stochastic signals $x(t)$ and $z(t)$, obtained from two spatially separated sensors (*e.g.* optical, capacitive, acoustic, thermal or scintillation), can be described using the following relationships:

$$x(t) = s(t) + n_1(t), \quad (1)$$

$$z(t) = c \cdot s(t - \tau_0) + n_2(t), \quad (2)$$

where: $s(t)$ – stationary random signal with frequency band B and Gaussian PDF $N(0, \sigma_s)$, c – constant factor; τ_0 – transportation time delay; $n_1(t)$, $n_2(t)$ – white noises with PDFs: $N(0, \sigma_{n1})$, $N(0, \sigma_{n2})$ and incoherent with the signal $s(t)$ and each other.

Formulas (1) and (2) are hereinafter referred to as Model I.

The autocorrelation function and the one-sided power spectral density of signal $s(t)$ are described by the equations:

$$R_{ss}(\tau) = KB \left(\frac{\sin 2\pi B\tau}{2\pi B\tau} \right), \quad (3)$$

$$G_{ss}(f) = \begin{cases} K & 0 \leq f \leq B \\ 0 & f > B \end{cases}, \quad (4)$$

where K denotes the constant value of the power spectral density $G_{ss}(t)$ in frequency band B .

For the signal models (1) and (2) the following relationships take place:

$$\sigma_x^2 = R_{xx}(0) = \sigma_s^2 + \sigma_{n_1}^2, \quad (5)$$

$$\sigma_z^2 = R_{zz}(0) = c^2 \sigma_s^2 + \sigma_{n_2}^2, \quad (6)$$

where σ_x and σ_z represent the standard deviations of the signals $x(t)$ and $z(t)$.

The SNR for models (1) and (2) can be defined as follows: $SNR_x = (\sigma_s/\sigma_{n_1})^2$ for $x(t)$ and $SNR_z = (\sigma_s/\sigma_{n_2})^2$ for $z(t)$.

In some cases of TDE, the signal models assume that $n_1(t) = n_2(t) = n(t)$. Then the disturbances $n_1(t)$ and $n_2(t)$ have identical normal PDFs $N(0, \sigma_n)$, but these are independent waveforms. In this case, formulas (1) and (2) will take the form:

$$x(t) = s(t) + n(t), \quad (7)$$

$$z(t) = c \cdot s(t - \tau_0) + n(t), \quad (8)$$

Dependencies (7) and (8) will hereinafter be referred to as Model II. The SNR for this model can be written as: $SNR_x = SNR_z = SNR = (\sigma_s/\sigma_n)^2$.

In the transport of solids, simple signal models (9) and (10) are often adopted:

$$x(t) = s(t), \quad (9)$$

$$z(t) = c \cdot x(t - \tau_0) + n(t). \quad (10)$$

This model results from the simplification of (1) and (2), for $n_1(t) = 0$ and $n_2(t) = n(t)$. Formulas (9) and (10) will hereafter be referred to as Model III. The signal-to-noise ratio for this is: $SNR = (\sigma_s/\sigma_n)^2$.

3. TDE by the use of CCF

3.1. Cross-correlation method

For signals $x(t)$ and $z(t)$, CCF $R_{xz}(\tau)$ can be given as follows [19, 41]:

$$R_{xz}(\tau) = E[(x(t)z(t + \tau))] = cR_{ss}(\tau - \tau_0), \quad (11)$$

where τ is the time delay and $E[\cdot]$ means the expected value.

For $\tau = \tau_0$, the function $R_{xz}(\tau)$ assumes a maximum value (Fig. 1). The transportation time delay can thus be designated as:

$$\tau_0 = \arg\{\max R_{xz}(\tau)\} = \arg\{R_{xz}(\tau_0)\}. \quad (12)$$

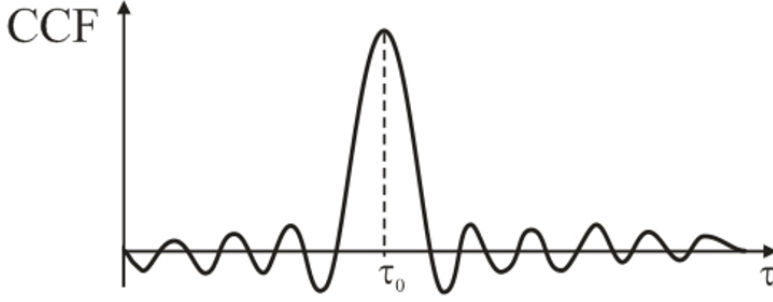


Fig. 1. Principle of TDE using CCF.

The normalized CCF is expressed as

$$\rho_{xz}(\tau) = \frac{R_{xz}(\tau)}{\sqrt{R_{xx}(0)R_{zz}(0)}} = \frac{cR_{ss}(\tau - \tau_0)}{\sigma_x \sigma_z} \quad (13)$$

and for $\tau = \tau_0$:

$$\rho_{xz}(\tau_0) = \frac{cR_{ss}(0)}{\sigma_x \sigma_z} = \frac{c\sigma_s^2}{\sigma_x \sigma_z}. \quad (14)$$

The dependencies of $\rho_{xz}(\tau_0)$ on the SNR for the analysed signal Models I, II and III are given in the paper [28].

In the currently used digital methods of analysis, the signals are sampled and quantized. Then the direct CCF estimator can be described using the formula:

$$\hat{R}_{xz}(\tau_m) = \frac{1}{N_{\text{total}}} \sum_{n=0}^{N_{\text{total}}-1} x_q(t_n) z_q(t_n + \tau_m), \quad m = 0, 1 \rightleftharpoons \dots M-1, \quad (15)$$

where: N_{total} is the number of samples, $x_q(n)$ and $z_q(n)$ are sampled and quantized signals $x(t)$ and $z(t)$, respectively. Equation (15) is valid for the number of signal samples z_q equal to $N_{\text{total}} + M$.

In [42], it was shown that when analysing random signals with a Gaussian PDF, the influence of quantization on the accuracy of the CCF estimation is negligible already for several levels of quantization. Thus, in further consideration, a negligible value of the quantization errors can be assumed.

In [28, 29], it was shown that the variance of the CCF estimator for $\tau = \tau_0$ is equal to:

$$\sigma^2 \left[\hat{R}_{xz}(\tau_0) \right] \approx \frac{\sigma_x^2 \sigma_z^2}{N_{\text{total}}} \left[1 + \rho_{xz}^2(\tau_0) \right]. \quad (16)$$

If CCF estimation is carried out with uncorrelated samples with the division of N_{total} into a number of N_d segments, then on the basis of (16) the relative standard uncertainty of the estimator can be represented by the formula:

$$u_{\text{rel}} \left[\hat{R}_{xz}(\tau_0) \right] = \frac{\sigma \left[\hat{R}_{xz}(\tau_0) \right]}{R_{xz}(\tau_0)} \approx \left\{ \frac{[1 + \rho_{xz}^2(\tau_0)]}{NN_d} \right\}^{1/2}, \quad (17)$$

where: N – the number of samples in a segment.

3.2. Standard uncertainty of the TDE using CCF

In the article [28], the formula for the variance of the time delay for CCF was derived in the following form:

$$\sigma^2(\tau_0)_{\text{CCF}} = \sqrt{\frac{3}{4}} \frac{1}{(\pi B)^2} \frac{\sigma \left[\widehat{R}_{xz}(\tau_0) \right]}{R_{xz}(\tau_0)}. \quad (18)$$

Based on (17) and (18), we obtain the formula for the standard uncertainty of the τ_0 obtained using CCF:

$$u(\tau_0)_{\text{CCF}} = \sigma(\tau_0)_{\text{CCF}} = \frac{1}{\pi B} \sqrt{\frac{3}{4}} \left\{ u_{\text{rel}} \left[\widehat{R}_{xz}(\tau_0) \right] \right\}^{1/2}. \quad (19)$$

In [28], it was shown that the standard uncertainties of the CCF based time delay estimation for models I, II, and III can be represented by the formulas:

– Model I

$$u(\tau_0)_{\text{CCF}} = \frac{1}{\pi B} \sqrt{\frac{3}{4}} \left\{ \frac{1}{\sqrt{NN_d}} \left[2 + \frac{1}{\text{SNR}_x} + \frac{1}{c^2} \left(\frac{1}{\text{SNR}_z} + \frac{1}{\text{SNR}_x} \frac{1}{\text{SNR}_z} \right) \right]^{1/2} \right\}^{1/2}, \quad (20)$$

– Model II

$$u(\tau_0)_{\text{CCF}} = \frac{1}{\pi B} \sqrt{\frac{3}{4}} \left\{ \frac{1}{\sqrt{NN_d}} \left[2 + \frac{1}{\text{SNR}} \left(1 + \frac{1}{c^2} \right) + \frac{1}{c^2 \text{SNR}^2} \right]^{1/2} \right\}^{1/2}, \quad (21)$$

– Model III

$$u(\tau_0)_{\text{CCF}} = \frac{1}{\pi B} \sqrt{\frac{3}{4}} \left\{ \frac{1}{\sqrt{NN_d}} \left[2 + \frac{1}{c^2 \text{SNR}} \right]^{1/2} \right\}^{1/2}. \quad (22)$$

4. Application of the phase of CSDF to TDE

4.1. Single-point phase method

TDE in the frequency domain is most often implemented by using the CSDF phase. In practice, only one-sided CSDF can be designated for the frequency range $0 < f < \infty$. This is a complex quantity described as follows [19]:

$$\underline{G}_{xz}(f) = 2 \int_{-\infty}^{\infty} R_{xz}(\tau) \cdot e^{-j2\pi f \tau} d\tau. \quad (23)$$

CSDF (23) can be represented in polar form:

$$\underline{G}_{xz}(f) = |G_{xz}(f)| e^{-j\Phi_{xz}(f)}, \quad 0 \leq f < \infty. \quad (24)$$

The phase of CSDF $\Phi_{xy}(f)$ for non-dispersive media is a linear function of the frequency f . For $\tau = \tau_0$ we obtain:

$$\Phi_{xy}(f) = \arctan \left\{ \frac{\text{Im} \left[\underline{G}_{-xz}(f) \right]}{\text{Re} \left[\underline{G}_{-xz}(f) \right]} \right\} = 2\pi \cdot f \cdot \tau_0. \quad (25)$$

Determination of the time delay from the CSDF phase is possible for the selected frequency f_0 (single-point method) or using a linear approximation of the phase characteristic in the selected frequency range (multipoint method) [19, 21, 31–33]. Next, the single-point method is analysed.

Based on (25), we obtain for the selected frequency $f = f_0$ (Fig. 2):

$$\tau_0 = \frac{\Phi_{xz}(f_0)}{2\pi f_0}. \quad (26)$$

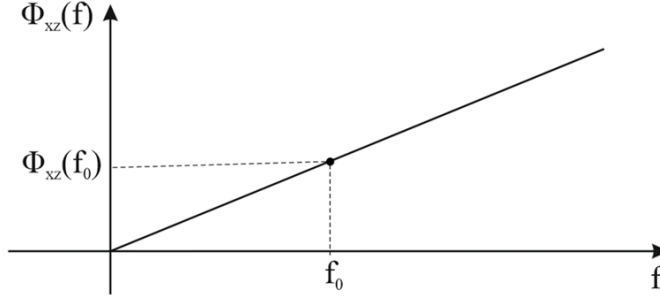


Fig. 2. Principle of the single-point CSDF phase method.

Normalized CSDF is known as the coherence function $\gamma_{xz}(f)$. The practical approach involves utilizing the squared coherence function:

$$\gamma_{xz}^2(f) = \frac{|G_{xz}(f)|^2}{G_{xx}(f) \cdot G_{zz}(f)}, \quad (27)$$

where $G_{xx}(f)$ and $G_{zz}(f)$ are autospectral density functions of signals $x(t)$ and $z(t)$, respectively.

The condition $0 \leq \gamma_{xz}^2(f) \leq 1$ is fulfilled by the $\gamma_{xz}^2(f)$.

The coherence functions for the models of signals I, II, and III discussed describe the equations:

– for Model I:

$$\begin{aligned} \gamma_{xz}^2(f) &= \left[1 + \frac{\sigma_{n1}^2}{\sigma_s^2} + \frac{\sigma_{n2}^2}{c^2 \sigma_s^2} + \frac{\sigma_{n1}^2}{c^2 \sigma_s^2} \frac{\sigma_{n2}^2}{\sigma_s^2} \right]^{-1} \\ &= \left[1 + \frac{1}{SNR_x} + \frac{1}{c^2} \left(\frac{1}{SNR_z} + \frac{1}{SNR_x} \frac{1}{SNR_z} \right) \right]^{-1}, \end{aligned} \quad (28)$$

– for Model II:

$$\gamma_{xz}^2(f) = \left[1 + \frac{\sigma_n^2}{\sigma_s^2} \left(1 + \frac{1}{c^2} \right) + \frac{\sigma_n^4}{c^2 \sigma_s^4} \right]^{-1} = \left[1 + \frac{1}{SNR} \left(1 + \frac{1}{c^2} \right) + \frac{1}{c^2 SNR^2} \right]^{-1}, \quad (29)$$

– for Model III:

$$\gamma_{xz}^2(f) = \left[1 + \left(\frac{\sigma_n}{c \sigma_s} \right)^2 \right]^{-1} = \left[1 + \frac{1}{c^2 SNR} \right]^{-1}. \quad (30)$$

The dependence (28) graphs are shown in Fig. 3, and the (29) and (30) graphs are presented in Fig. 4.

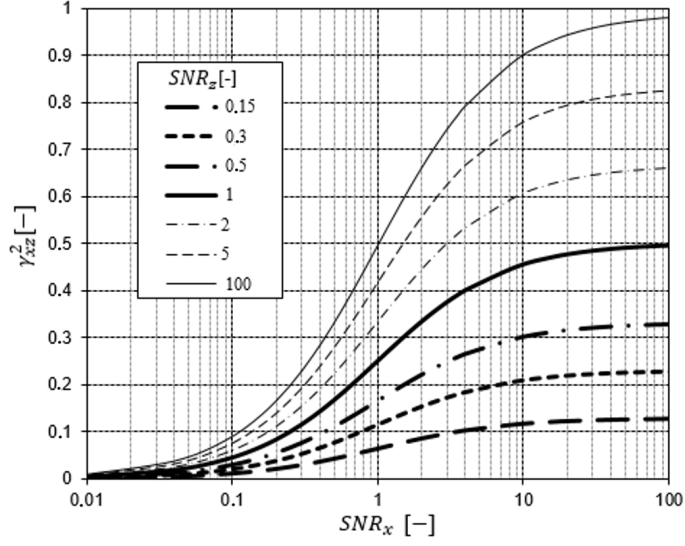


Fig. 3. Plots of the relationship (28) for various SNR_z values and $c = 1$.

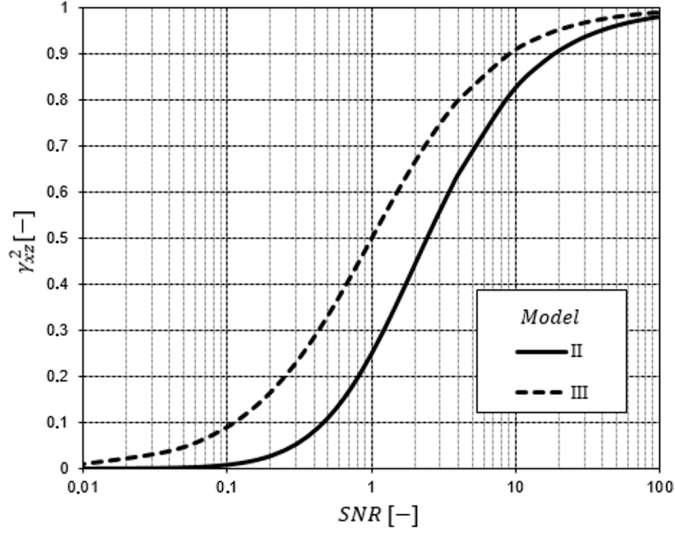


Fig. 4. Plots of the relationships (29) and (30) for various SNR values and $c = 1$.

4.2. Standard uncertainty of the TDE from phase CSDF

The accuracy of the TDE based on CSDF depends on the variance of the phase estimator, which is given by [19]:

$$\sigma^2 [\tilde{\Phi}_{xz}(f)] \approx \frac{1 - \gamma_{xz}^2(f)}{2N_d \gamma_{xz}^2(f)}, \quad (31)$$

where $\tilde{\Phi}_{xz}(f)$ is the smoothed CSDF phase estimator determined using a phase unwrapping and averaging procedure, *e.g.* the Welch method [19, 43].

[31] shows that the standard uncertainty of the TDE when utilizing the CSDF phase for $f = f_0$ is given by the following:

$$u[\tau_0]_{\text{CSDF}} = \sigma[\tau_0]_{\text{CSDF}} \approx \frac{1}{2\pi f_0} \left\{ \frac{1 - \gamma_{xz}^2(f_0)}{2N_d \gamma_{xz}^2(f_0)} \right\}^{1/2}. \quad (32)$$

Taking into account the coherence functions (28)–(30) for the signal models analysed, we obtain for $f = f_0$:

– for Model I:

$$u[\tau_0]_{\text{CSDF}} \approx \frac{1}{2\pi f_0} \left\{ \frac{1}{2N_d} \left[\frac{1}{\text{SNR}_x} + \frac{1}{c^2} \left(\frac{1}{\text{SNR}_z} + \frac{1}{\text{SNR}_x} \frac{1}{\text{SNR}_z} \right) \right] \right\}^{1/2}, \quad (33)$$

– for Model II:

$$u[\tau_0]_{\text{CSDF}} \approx \frac{1}{2\pi f_0} \left\{ \frac{1}{2N_d} \left[\frac{1}{\text{SNR}} \left(1 + \frac{1}{c^2} \right) + \frac{1}{c^2 \text{SNR}^2} \right] \right\}^{1/2}, \quad (34)$$

– for Model III:

$$u[\tau_0]_{\text{CSDF}} \approx \frac{1}{2\pi f_0} \left\{ \frac{1}{2N_d c^2 \text{SNR}} \right\}^{1/2}. \quad (35)$$

5. Comparison of standard uncertainties of TDE using the CSDF-phase and CCF methods

After a comparison of (19) and (32) we obtain:

$$\frac{u(\tau_0)_{\text{CSDF}}}{u(\tau_0)_{\text{CCF}}} = \frac{B}{2f_0} \sqrt[4]{\frac{4}{3}} \left\{ \frac{1}{N_d} \left[\frac{1}{\gamma_{xz}^2(f_0)} - 1 \right] \frac{1}{u_{\text{rel}}[\widehat{R}_{xz}(\tau_0)]} \right\}^{1/2}. \quad (36)$$

Taking quotients of formulas (20) and (33), (21) and (34), and (22) and (35) we get the following dependencies for the signal models:

– Model I:

$$\frac{u(\tau_0)_{\text{CSDF}}}{u(\tau_0)_{\text{CCF}}} = \frac{B}{f_0} \frac{1}{\sqrt[4]{48}} \left\{ \frac{N \left[\frac{1}{\text{SNR}_x} + \frac{1}{c^2 \text{SNR}_z} \left(1 + \frac{1}{\text{SNR}_x} \right) \right]^2}{N_d \left(2 + \frac{1}{\text{SNR}_x} + \frac{1}{c^2 \text{SNR}_z} \left(1 + \frac{1}{\text{SNR}_x} \right) \right)} \right\}^{1/4}, \quad (37)$$

– Model II:

$$\frac{u(\tau_0)_{\text{CSDF}}}{u(\tau_0)_{\text{CCF}}} = \frac{B}{f_0} \frac{1}{\sqrt[4]{48}} \left\{ \frac{N \left[\frac{1}{\text{SNR}} \left(1 + \frac{1}{c^2} \right) + \frac{1}{c^2 \text{SNR}^2} \right]^2}{N_d \left(2 + \frac{1}{\text{SNR}} \left(1 + \frac{1}{c^2} \right) + \frac{1}{c^2 \text{SNR}^2} \right)} \right\}^{1/4}, \quad (38)$$

– Model III:

$$\frac{u(\tau_0)_{\text{CSDF}}}{u(\tau_0)_{\text{CCF}}} = \frac{B}{f_0} \frac{1}{\sqrt[4]{48}} \left\{ \frac{N}{N_d} \left[\frac{1}{c^2 \text{SNR} (2c^2 \text{SNR} + 1)} \right] \right\}^{1/4}. \quad (39)$$

The analysis of formula (32) shows that possible large values of f_0 should be selected. The maximum frequency f_0 can be equal to the signal frequency band B . The plots of the formulas (37), (38) and (39) for $B/f_0 = 1$ and for $c = 1$ are shown in Figs. 5, 6 and 7.

For small SNRs, the two in the denominators of (37) and (38) can be omitted, and we obtain, respectively, for models I and II:

$$\frac{u(\tau_0)_{\text{CSDF}}}{u(\tau_0)_{\text{CCF}}} = \frac{B}{f_0} \frac{1}{\sqrt[4]{48}} \left\{ \frac{N}{N_d} \left[\frac{1}{\text{SNR}_x} + \frac{1}{c^2 \text{SNR}_z} \left(1 + \frac{1}{\text{SNR}_x} \right) \right] \right\}^{1/4}, \quad (40)$$

$$\frac{u(\tau_0)_{\text{CSDF}}}{u(\tau_0)_{\text{CCF}}} = \frac{B}{f_0} \frac{1}{\sqrt[4]{48}} \left\{ \frac{N}{N_d} \left[\frac{1}{\text{SNR}} \left(1 + \frac{1}{c^2} \right) + \frac{1}{c^2 \text{SNR}^2} \right] \right\}^{1/4}. \quad (41)$$

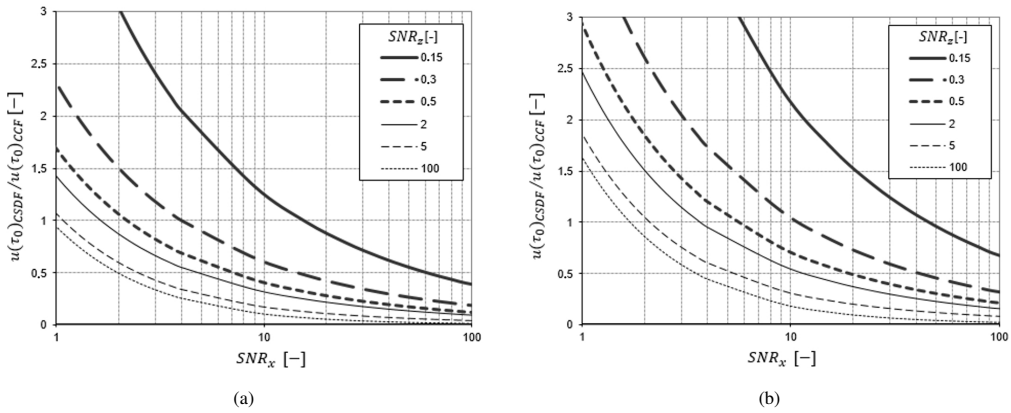


Fig. 5. Plots of the dependence (37) with $N/N_d = 110$ (a) and $N/N_d = 1,000$ (b), for various values of SNR_z , $B/f_0 = 1$, and $c = 1$.

Based on the analysis of formulas (37) to (39) and the plots shown in Figs. 5 to 7, it can be concluded that, in general, the single-point phase method gives smaller values of the standard uncertainty of the TDE than the cross-correlation for SNR values greater than unity. SNR limits (for which $u(\tau_0)_{\text{CSDF}} = u(\tau_0)_{\text{CCF}}$) depend on the adopted signal model and the analysis parameters: N , N_d , B , f_0 . In the case of Model I, for given values of SNR_x , B/f_0 , and c , the values of SNR_z and the N/N_d ratio are important. Figure 5 shows the influence of the SNR_z values on the ratio of the number of samples in a segment N to the number of segments N_d for two selected cases: when $N/N_d = 110$ (Fig. 5a) and when $N/N_d = 1,000$ (Fig. 5b). It should be noted that the ratio N/N_d can take different values depending on the available number of samples N_{total} . For example, for $N_{\text{total}} = 100,000$, the chosen values of N , N_d , and N/f_0 (lub N/N_d) are summarized in Table 1. From the perspective of minimizing the variance of the CCF and CSDF phase estimators, the total number of samples N_{total} and the number of segments N_d should be as large as possible, which follows directly from (16) and (31). On the other hand, the number of samples in a segment N should also be large. In practice, it is therefore necessary to find a compromise solution.

Figure 5 allows determining the ranges of SNR_x and SNR_z values for which the standard uncertainty of time delay estimation from the CSDF phase will be smaller than for the CCF. For example, from Fig. 5a, one can infer that the uncertainty $u(\tau_0)_{\text{CSDF}}$ will be smaller than $u(\tau_0)_{\text{CCF}}$ if the SNR_x value is greater than 1.3 and $\text{SNR}_z > 0.3$. In the case of $\text{SNR}_x = 1$, the SNR_z value for this situation would need to be almost one hundred.

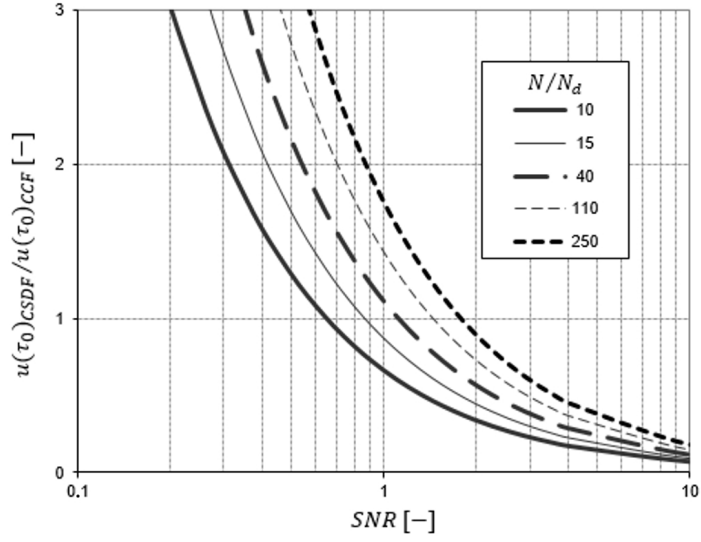


Fig. 6. Plots of the dependence (38) for various values of N/N_d , $B/f_0 = 1$, and $c = 1$

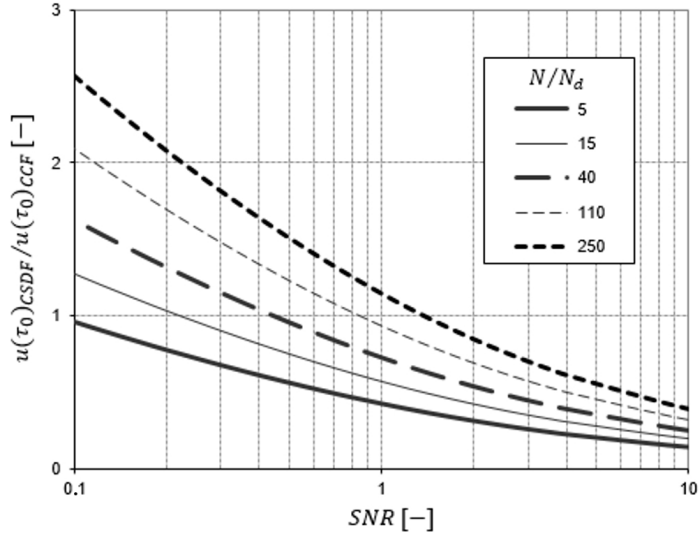


Fig. 7. Plots of the dependence (39) for various values of N/N_d , $B/f_0 = 1$, and $c = 1$

Table 1. Selected values of N and N_d for $N_{\text{total}} = 100,000$.

N	4,000	5,000	6,250	10,000
N_d	25	20	16	10
N/N_d	160	250	444.44	1,000

Figures 6 and 7 show the dependencies $u(\tau_0)_{\text{CSDF}}/u(\tau_0)_{\text{CCF}} = f(\text{SNR})$ for signal models II and III, respectively. For these cases, the parameter is the ratio N/N_d . For Model II, for $c = 1$ and $B/f_0 = 1$, the limiting SNR values (for which $u(\tau_0)_{\text{CSDF}} = u(\tau_0)_{\text{CCF}}$) range from 0.63 (for $N/N_d = 10$) to 1 (for $N/N_d = 250$). In the case of signal model III, the SNR limits range from 0.1 (for $N/N_d = 5$) to 1.03 (for $N/N_d = 250$).

In practice, the signal-to-noise ratio in measurement signals is usually unknown, but signal models can be determined. In such cases, the SNR can be assessed using the experimentally determined normalized cross-correlation and the relationships $\rho_{xz}(\tau_0) = f(\text{SNR})$ for the adopted signal models. These relationships for signal models I-III are provided in article [28].

6. Conclusions

This paper compares the standard uncertainties of the time delay estimation using the methods of cross-correlation and phase cross-spectral density (single-point method). Analyses were performed for three models of mutually delayed stochastic measurement signals with random disturbances. It was found that in the range of small signal-to-noise ratio values ($\text{SNR} < 1$), the time delay estimation from the cross-correlation function is more accurate, while in the range of higher SNR, lower uncertainty values of TDE can be obtained for the phase method. Specific SNR values for which the phase method is more accurate depend on the adopted models and parameters of the signals and on the parameters of the analysis. The advantage of the single-point phase method is the ability to determine the time delay for the selected frequency f_0 , which lies in the band of measurement signals. This is important in dispersive media where the CSDF phase is not a linear function of frequency. The new results obtained in this work are relations (37)–(39) that allow for comparing the standard uncertainties of the CCF and single-point CSDF estimators for the three assumed signal models.

References

- [1] Yao, S., Meng, Q., Chen, C., & Tariq, I. (2022). A DPTF algorithm for the time-delay estimation in the reflected environment. *Digital Signal Processing*, 127, 103534. <https://doi.org/10.1016/j.dsp.2022.103534>
- [2] Costa-Júnior, J. F., Cortela, G. A., Maggi, L. E., Rocha, T. F., Pereira, W. C., Costa-Felix, R. P., & Alvarenga, A. V. (2018). Measuring uncertainty of ultrasonic longitudinal phase velocity estimation using different time-delay estimation methods based on cross-correlation: Computational simulation and experiments. *Measurement*, 122, 45–56. <https://doi.org/10.1016/j.measurement.2018.01.073>
- [3] Uchendu, N., Muggleton, J. M., & White, P. R. (2025). Cepstral method for time delay estimation in acoustic leak detection. *Mechanical Systems and Signal Processing*, 241, 113448. <https://doi.org/10.1016/j.ymssp.2025.113448>
- [4] Duda, K., & Marszałek, Z. (2024). Vehicle speed determination with inductive-loop technology and fast and accurate fractional time delay estimation by DFT. *Metrology and Measurement Systems*, Vol. 31 (2024) No. 4, pp. 781–796. <https://doi.org/10.24425/mms.2024.152048>
- [5] Waschburger, R., & Galvão, R. K. H. (2012). Time delay estimation in discrete-time state-space models. *Signal Processing*, 93(4), 904–912. <https://doi.org/10.1016/j.sigpro.2012.10.017>

- [6] Bharathi, B. M. R., & Mohanty, A. (2019). Time delay estimation in reverberant and low SNR environment by EMD based maximum likelihood method. *Measurement*, 137, 655–663. <https://doi.org/10.1016/j.measurement.2019.01.096>
- [7] Assous, S., & Linnett, L. (2012). High resolution time delay estimation using sliding discrete Fourier transform. *Digital Signal Processing*, 22(5), 820–827. <https://doi.org/10.1016/j.dsp.2012.05.001>
- [8] Gedalyahu, K., & Eldar, Y. C. (2010). Time-delay estimation from low-rate samples: a union of subspaces approach. *IEEE Transactions on Signal Processing*, 58(6), 3017–3031. <https://doi.org/10.1109/tsp.2010.2044253>
- [9] Blok, E. (2002). Classification and evaluation of discrete subsample time delay estimation algorithms. In *14th International Conference on Microwaves, Radar and Wireless Communications. MIKON – 2002* (Vol. 3, pp. 764–767). <https://doi.org/10.1109/MIKON.2002.1017953>
- [10] Cobos, M., Antonacci, F., Comanducci, L., & Sarti, A. (2020). Frequency-sliding generalized cross-correlation: a sub-band time delay estimation approach. *IEEE/ACM Transactions on Audio Speech and Language Processing*, 28, 1270–1281. <https://doi.org/10.1109/taslp.2020.2983589>
- [11] Du, C., Yu, S., Yin, H., & Sun, Z. (2022). Microseismic time delay estimation method based on continuous wavelet. *Sensors*, 22(8), 2845. <https://doi.org/10.3390/s22082845>
- [12] Chen, J., Benesty, J., & Huang. (2006). Time delay estimation in room acoustic environments: an overview. *EURASIP Journal on Advances in Signal Processing*, 2006(1). <https://doi.org/10.1155/asp/2006/26503>
- [13] Kousiopoulos, G., Papastavrou, G., Kampelopoulos, D., Karagiorgos, N., & Nikolaidis, S. (2020). Comparison of time delay estimation methods used for fast pipeline leak localization in high-noise environment. *Technologies*, 8(2), 27. <https://doi.org/10.3390/technologies8020027>
- [14] Müller, T., Lauk, M., Reinhard, M., Hetzel, A., Lücking, C. H., & Timmer, J. (2003). Estimation of delay times in biological systems. *Annals of Biomedical Engineering*, 31(11), 1423–1439. <https://doi.org/10.1114/1.1617984>
- [15] Brennan, M., Gao, Y., & Joseph, P. (2007). On the relationship between time and frequency domain methods in time delay estimation for leak detection in water distribution pipes. *Journal of Sound and Vibration*, 304(1–2), 213–223. <https://doi.org/10.1016/j.jsv.2007.02.023>
- [16] Hanus, R., Zych, M., Petryka, L., & Świsulski, D. (2014). Time delay estimation in two-phase flow investigation using the γ -Ray attenuation technique. *Mathematical Problems in Engineering*, 2014(1). <https://doi.org/10.1155/2014/475735>
- [17] Faerman, V., Avramchuk, V., Voevodin, K., Sidorov, I., & Kostyuchenko, E. (2022). Study of generalized phase spectrum time delay estimation method for source positioning in small room acoustic environment. *Sensors*, 22(3), 965. <https://doi.org/10.3390/s22030965>
- [18] Knapp, C., & Carter, G. (1976). The generalized correlation method for estimation of time delay. *IEEE Transactions on Acoustics Speech and Signal Processing*, 24(4), 320–327. <https://doi.org/10.1109/tassp.1976.1162830>
- [19] Bendat, J. S., & Piersol, A. G. (2010). Random data. In *Wiley Series in Probability and Statistics*. 4th ed. Wiley, New York, USA. <https://doi.org/10.1002/9781118032428>
- [20] Hertz, D., & Azaria, M. (1985). Time delay estimation between two phase shifted signals via generalized cross-correlation methods. *Signal Processing*, 8(2), 235–257. [https://doi.org/10.1016/0165-1684\(85\)90076-3](https://doi.org/10.1016/0165-1684(85)90076-3)

- [21] Piersol, A. (1981). Time delay estimation using phase data. *IEEE Transactions on Acoustics Speech and Signal Processing*, 29(3), 471–477. <https://doi.org/10.1109/tassp.1981.1163555>
- [22] Mosorov, V. (2006). Phase spectrum method for time delay estimation using twin-plane electrical capacitance tomography. *Electronics Letters*, 42(11), 630–632. <https://doi.org/10.1049/el:20060338>
- [23] Matassoni, M., & Svaizer, P. (2006). Efficient time delay estimation based on cross-power spectrum phase. In *14th European Signal Processing Conference*. <https://doi.org/10.5281/zenodo.39836>
- [24] Jacovitti, G., & Scarano, G. (1993). Discrete time techniques for time delay estimation. *IEEE Transactions on Signal Processing*, 41(2), 525–533. <https://doi.org/10.1109/78.193195>
- [25] Chen, J., Benesty, J., & Huang, Y. (2005). Performance of GCC- and AMDF-based time-delay estimation in practical reverberant environments. *EURASIP Journal on Advances in Signal Processing*, 2005(1). <https://doi.org/10.1155/asp.2005.25>
- [26] Hanus, R., Kowalczyk, A., Szlachta, A. & Chorzępa, R. (2018). Application of conditional averaging to time delay estimation of random signals. *Measurement Science Review*, 18(4), 130–137. <https://doi.org/10.1515/msr-2018-0019>
- [27] Kowalczyk, A., Hanus, R., & Szlachta, A. (2011). Investigation of the statistical method of time delay estimation based on conditional averaging of delayed signal. *Metrology and Measurement Systems*, Vol. 18 (2011), No. 2, pp. 335–342. <https://doi.org/10.2478/v10178-011-0015-3>
- [28] Hanus, R. (2019). Time delay estimation of random signals using cross-correlation with Hilbert transform. *Measurement*, 146, 792–799. <https://doi.org/10.1016/j.measurement.2019.07.014>
- [29] Bendat, J.S. (1985). *The Hilbert Transform and Applications to Correlation Measurements*. Brüel & Kjær, BT0008-11, Naerum, Denmark
- [30] Zhen, Z., & Zi-Qiang, H. (2005). The generalized phase spectrum method for time delay estimation. *Proc. of the IEEE International Conference on Acoustics, Speech, and Signal Processing ICASSP 084*, San Diego, CA, USA, 19–21 March 1984, 459–462. <https://doi.org/10.1109/icassp.1984.1172709>
- [31] Hanus, R. (2003). Statistical error analysis of time delay measurement by using phase of cross-spectral density function. *Systems Analysis Modelling Simulation*, 43(8), 993–998. <https://doi.org/10.1080/0232929031000080632>
- [32] Hanus, R., Zych, M., & Golijanek-Jędrzejczyk, A. (2022). Measurements of dispersed phase velocity in two-phase flows in pipelines using gamma-absorption technique and phase of the cross-spectral density function. *Energies*, 15(24), 9526. <https://doi.org/10.3390/en15249526>
- [33] Bendat, J. S., & Piersol, A. G. (1993). *Engineering Applications of Correlation and Spectral analysis*. 2nd ed. Wiley, New York, USA.
- [34] Uchendu, N., Muggleton, J. M., & White, P. R. (2025a). Cross-spectral phase method for time delay estimation in acoustic leak detection. *Applied Acoustics*, 235, 110695. <https://doi.org/10.1016/j.apacoust.2025.110695>
- [35] Ma, Y., Gao, Y., Cui, X., Brennan, M. J., Almeida, F. C., & Yang, J. (2019). Adaptive phase transform method for pipeline leakage detection. *Sensors*, 19(2), 310. <https://doi.org/10.3390/s19020310>
- [36] Omologo, M., & Svaizer, P. (1997). Use of the crosspower-spectrum phase in acoustic event location. *IEEE Transactions on Speech and Audio Processing*, 5(3), 288–292. <https://doi.org/10.1109/89.568735>
- [37] Lindemann, M., Raethjen, J., Timmer, J., Deuschl, G., & Pfister, G. (2001). Delay estimation for cortico-peripheral relations. *Journal of Neuroscience Methods*, 111(2), 127–139. [https://doi.org/10.1016/S0165-0270\(01\)00436-8](https://doi.org/10.1016/S0165-0270(01)00436-8)

- [38] Witham, C. L., Riddle, C. N., Baker, M. R., & Baker, S. N. (2011). Contributions of descending and ascending pathways to corticomuscular coherence in humans. *The Journal of Physiology*, 589(15), 3789–3800. <https://doi.org/10.1113/jphysiol.2011.211045>
- [39] Timmer, J., Lauk, M., Pfleger, W., & Deuschl, G. (1998). Cross-spectral analysis of physiological tremor and muscle activity. *Biological Cybernetics*, 78(5), 349–357. <https://doi.org/10.1007/s004220050439>
- [40] Hanus, R. (2008). Statistical error comparison of time delay estimation using cross-correlation function and phase of cross-spectral density function. *Przegląd Elektrotechniczny*, 84(12), 301-303. (in Polish)
- [41] Beck, M.S. & Płaskowski, A. (1987). *Cross-correlation Flowmeters, Their Design and Application*. Adam Hilger, Bristol, UK.
- [42] Lal-Jadziak, J. (2001). *Influencing Accuracy in Correlation Measurements*. Wydawnictwo Politechniki Zielonogórskiej, Zielona Góra, Poland. (in Polish)
- [43] Lyons, R.G. (2011). *Understanding Digital Signal Processing*. 3rd ed., Prentice Hall, New York, USA.



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