

A Hybrid Fuzzy-Statistical Framework for Process Capability Analysis under Non-Normality and Asymmetric Tolerances

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Abstract

This paper proposes a methodology for estimating process capability indices (PCIs) when data deviate from normality and the tolerances of quality characteristics are both asymmetric and imprecise. A fuzzy logic approach was applied to address uncertainty in tolerances. The methodology has five steps: (1) Fit a heavy-tailed distribution to the process data, (2) Select the best-fitting distribution, (3) Generate random samples from the selected distribution, (4) Construct triangular fuzzy numbers using the distribution parameters, and (5) Calculate the PCIs and analyze the results. The proposed methodology offers a more accurate assessment of process capability, benefiting industries that manage non-normal data and asymmetric specifications under uncertain conditions. In the applied case, the proposed methodology reduced the relative error in estimating the process capability index value by an average of 49% compared to the traditional method. The primary contribution of this paper is the integration of fuzzy logic with PCIs estimation to address imprecision and asymmetry, thereby enhancing process capability analysis under conditions closer to the reality of process quality control.

Keywords

Asymmetric fuzzy tolerances, Fuzzy logic, Non-normal data, Process capability indices, Statistical quality control.

Introduction

Process Capability Indices (PCIs) quantify how well a process meets its specifications, regarding the centering and variability (English & Taylor, 1993). Traditional PCI estimations assume that the data follow a normal distribution and are accurate. Moreover, these calculations typically consider processes with symmetric tolerances, where the target specification value (T) is centered between the Lower Specification Limit (LSL) and the Upper Specification Limit (USL) (Montgomery, 2009). Capability analysis involves com-

paring actual process variation to specified limits, as well as determining how closely it aligns with the specification value (Montgomery, 2009; Gutiérrez & De La Vara, 2009).

Figure 1 provides a schematic overview of the various approaches used for estimating PCIs. When the assumptions of normality and symmetry are not met, traditional PCI estimation techniques face limitations in both their application and interpretation. To address these challenges, several modifications to standard PCIs have been proposed, as outlined by Kotz & Johnson (1993). Common approaches to handling non-normal data involve applying power transformations (Box & Cox, 1982), or converting an asymmetric distribution into a normal (Somerville & Montgomery, 1996). Alternatively, Clements (1989) proposed a method that addresses non-normal data by determining non-normal percentiles and modifying traditional PCIs using the family of Pearson curves.

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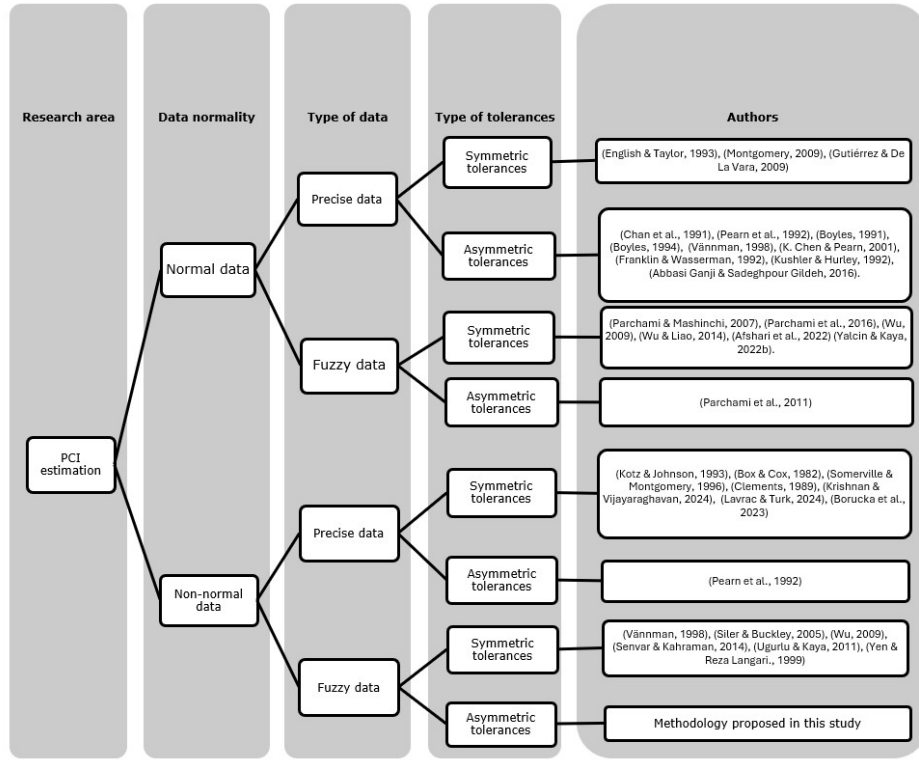


Fig. 1. Overview of PCI estimation approaches for different data characteristics.

Literature review

PCI estimation for normal, precise data with symmetric or asymmetric tolerances

This section outlines the methodologies for estimating various PCIs, including the potential process capability index (C_p), the upper specification capability index (C_{pu}), the lower specification capability index (C_{pl}), and the actual process capability index (C_{pk}).

$$C_p = \frac{USL - LSL}{6\sigma} \quad (1)$$

$$C_{pu} = \frac{USL - \mu}{3\sigma} \quad (2)$$

$$C_{pl} = \frac{\mu - LSL}{3\sigma} \quad (3)$$

$$C_{pk} = \min(C_{pu}, C_{pl}) \quad (4)$$

Since these PCIs do not consider process loss due to deviations from the target value (T), the indices C_{pm} and C_{pmk} were developed (Chan et al., 1991; Pearn et al., 1992). The C_{pm} index considers the distance between the process mean and the target value, making it more sensitive to deviations than

C_p and C_{pk} . Similarly, the C_{pmk} index combines the metrics of standard PCIs and is also more sensitive to process deviations (Boyles, 1991).

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} \quad (5)$$

$$C_{pmk} = \min \left\{ \frac{USL - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - LSL}{3\sqrt{\sigma^2 + (\mu - T)^2}} \right\} \quad (6)$$

In the previous equations, μ is the process mean; and σ denotes the standard deviation. For processes with bilateral specifications, indices C_p and C_{pk} are applied, while for unilateral specifications, C_{pl} and C_{pu} are used, depending on the situation.

Asymmetric tolerances occur when customers are less accepting of deviations from the target specification in one direction than in the other. In line with this, manufacturers may encounter customers willing to extend tolerance only on one side of the target (Boyles, 1994). Asymmetric tolerances may also occur when initial tolerances are symmetric, but the process distribution is skewed, allowing suppliers to set specifications based on this distribution. Additionally, these tolerances may arise when data transformations are performed to achieve approximate normality.

In the specialized literature, several approaches address the issues of data normality, data accuracy, and asymmetric tolerances. These methods involve replacing the true specification limits ($T - D_l, T + D_u$) with symmetric limits ($T - d^*, T + d^*$), where $D_l = T - LSL$, $D_u = USL - T$, and $d^* = \min\{D_l, D_u\}$, and subsequently applying the standard definitions of C_p , C_{pk} , and C_{pmk} (Vännman, 1998; Chen & Pearn, 2001). However, although the true specification limits are replaced with ersatz symmetric limits ($T - d', T + d'$), where $d' = \frac{D_l + D_u}{2}$, this can lead to over- or underestimation of process capability, depending on the relationship between the process mean (μ) and the target (T) (Franklin & Wasserman, 1992; Kushler & Hurley, 1992).

Boyles introduced a smoothing function,

$$S(x, y) = \frac{\Phi^{-1} \left\{ \frac{\Phi(x)}{2} + \frac{\Phi(y)}{2} \right\}}{3},$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. When applied to C_{pmk} , this function yields a new index, S_{pmk} , which more accurately represents process performance and is better suited for general use (Boyles, 1994). Additionally, Abbasi & Sadeghpour (2016) proposed factors that adjust the capability assessment for processes deviating toward asymmetric specification limits, providing a more realistic evaluation of actual process characteristics.

Chen & Pearn (2001) demonstrated that these new capability indices for processes with asymmetric tolerances are more accurate and outperform traditional indices in evaluating both process performance and potential.

PCI estimation for normal, fuzzy data with symmetric or asymmetric tolerances

In an industrial environment, the random variable of a process may be recorded as an imprecise (fuzzy) value, and the specification limits may lack accuracy. In such cases, traditional methods become inadequate, as their estimates do not accurately reflect the true performance of the process.

Fuzzy logic provides an alternative approach to address imprecision in PCI estimation and allows modeling of vagueness and uncertainty in a probabilistic or random manner (Zadeh, 1965; Klir & Yuan, 1995; Siler & Buckley, 2005). Fuzzy logic uses a function to represent the membership degree of an element to a set, and an element can belong to multiple sets with different membership degrees. Fuzzy models allow the treatment of imprecision and uncertainty of systems when the boundaries between sets are not strictly defined (Jang et al., 1997; Kasabov, 1998; Zimmermann, 2001; Ross, 2004; Hesamian & Akbari, 2018).

Several works have been developed for estimating fuzzy PCIs. Parchami & Mashinchi (2007) proposed a fuzzy PCI for normal distributions considering parameters with precise data and fuzzy specification limits. Also, Parchami et al. (2016) proposed modified classical PCIs (C_p , C_{pk} , and C_{pm}) for processes with imprecise specifications and data.

Wu (2009) proposed to construct triangular confidence intervals to estimate the index C_{pk} and to use rules based on both fuzzy critical values and fuzzy p-values to assess process performance. Later, Wu & Liao (2014) proposed a fuzzy nonlinear programming method to assess and rank process performance with imprecise data.

Afshari et al. (2022) analyzed the impact of measurement errors on multivariate process capability assessment using PCIs. They developed a linear error model and used traditional methods along with fuzzy estimation techniques for the treatment of imprecise data. Similarly, Yalcin & Kaya (2022b) researched process capability assessment using fuzzy process averages and specification limits to estimate the C_p and C_{pk} indices. Finally, Parchami et al. (2011) developed a generalized C_{pm} that considers asymmetric tolerances and fuzzy specification limits, and Yalcin & Kaya (2022a) applied Pythagorean fuzzy set to address uncertainty in data.

PCI estimation for non-normal, precise data with symmetric or asymmetric tolerances

Some modifications to traditional PCI estimation methods have been proposed to overcome their limitations (Kotz & Johnson, 1993), including the use of non-normal data transformations (Box & Cox, 1982), and the conversion of asymmetric distributions to normal ones before applying traditional PCIs (Somerville & Montgomery, 1996). Wang et al. (2021) proposed the Y_p index to handle asymmetric specifications and non-normal processes, by calculating confidence intervals using bootstrap methods. Another approach to dealing with non-normal data is to determine non-normal percentiles and use the family of Pearson curves (Clements, 1989). Building on this percentile-based approach, Pearn & Chen (1995) extended the method to Pearsonian populations, while Pearn & Chen (1997) further generalized it to obtain capability indices such as C_{Npmk} for non-normal distributions not restricted to the Pearson family. In the proposed formulation, C_{Npmk} can be written as in (7), preserving the logic of Clements while adapting the denominator to reflect target deviation and process asymmetry.

$$C_{Npmk} = \min \left\{ \frac{\frac{USL - M}{3\sqrt{\left[\frac{F_{99.865} - F_{0.135}}{6}\right]^2 + (M - T)^2}}}{\frac{M - LSL}{3\sqrt{\left[\frac{F_{99.865} - F_{0.135}}{6}\right]^2 + (M - T)^2}}} \right\} \quad (7)$$

Finally, recent work has used Box-Cox transformation and goodness-of-fit tests in process capability analysis with lognormal data, obtaining accurate estimates by applying the Shapiro-Wilk and Artificial Covariate methods (Krishnan & Vijayaraghavan, 2024). Similarly, Lavrac & Turk (2024) proposed a modified power transformation for estimating PCIs with non-normal data, which reduces skewness and improves their accuracy. In addition, Borucka et al. (2023) developed a methodology that employs multicriteria decision analysis to assess capability in non-normal data by minimizing the Jarque-Bera statistics to approximate the normality of the data and Karakaya (2024) proposed the use of quantiles and the Weibull distribution.

PCI estimation for non-normal, fuzzy data with symmetric or asymmetric tolerances

Although several methodologies for PCI estimation have been developed to address non-normal data, uncertainty, and symmetric tolerances (Senvar & Kahraman, 2014; Ugurlu & Kaya, 2011), there is still a lack of integrated approaches that evaluate these conditions simultaneously. Some works have used Monte Carlo simulation with fuzzy specification limits to evaluate the process capability with non-normal data, test models in processes with asymmetric tolerances and comparison between maximum likelihood, bootstrap and Markov chain Monte Carlo techniques with inverted exponential distributions under uncertainty (Iranmanesh et al., 2022; Chen & Yu, 2024; Hasaballah et al., 2024).

To achieve this, the authors perform process adjustments and calculate the necessary parameters using metaheuristic techniques. Additionally, fuzzy triangular numbers are optimally construct for the specification limits by applying a modified version of Clements method (Cogollo Flórez et al., 2017; Arteaga Sierra et al., 2017).

Materials & Methods

Let LSL , T , and USL be the lower specification limit, target value, and upper specification limit of the process quality characteristic, respectively. Additionally, let $x_{0.135}$, x_{50} , and $x_{99.865}$ denote their corresponding percentiles. Define $A = (a_1, a_2, a_3)$ as a Triangular Fuzzy Number (TFN), whose α -cut-off values are calculated as follows (Zadeh, 1965; Dutta et al., 2011):

$$A_\alpha = [a_1 + (a_2 - a_1)\alpha, a_3 + (a_2 - a_3)\alpha] = [A_l, A_r] \quad (8)$$

Here, A_l and A_r represent the left and right limits, respectively, of the corresponding interval at a specific α -level, where $\alpha \in [0, 1]$.

The equations for estimating the PCIs for non-normal data with asymmetric fuzzy tolerances ($C_{pmu}(\alpha)$, $C_{pml}(\alpha)$, and $C_{pmk}(\alpha)$) are defined as follows:

$$C_{pmu}(\alpha) = \frac{(USL - M)\alpha}{3\sqrt{\left(\frac{1}{9}(U_p - M)^2 + (M - T)^2\right)\alpha}} \quad (9)$$

$$= \frac{[usl_1 - m_3 + (usl_2 - m_2 - usl_1 + m_3)\alpha, usl_3 - m_1 + (usl_2 - m_2 - usl_3 + m_1)\alpha]}{3\sqrt{y_1 + (y_2 - y_1)\alpha}, 3\sqrt{y_3 + (y_2 - y_3)\alpha}}$$

$$= \left[\frac{usl_1 - m_3 + (usl_2 - m_2 - usl_1 + m_3)\alpha}{3\sqrt{y_3 + (y_2 - y_3)\alpha}}, \frac{usl_3 - m_1 + (usl_2 - m_2 - usl_3 + m_1)\alpha}{3\sqrt{y_1 + (y_2 - y_1)\alpha}} \right]$$

where,

$$y_1 = \frac{1}{9}(up_1^2 - 2up_1m_3 + m_3^2) + m_1^2 - 2m_1t_3 + t_3^2$$

$$y_2 = \frac{1}{9}(up_2^2 - 2up_2m_2 + m_2^2) + m_2^2 - 2m_2t_2 + t_2^2$$

$$y_3 = \frac{1}{9}(up_3^2 - 2up_3m_1 + m_1^2) + m_3^2 - 2m_3t_1 + t_1^2$$

$$C_{pml}(\alpha) = \frac{(M - LSL)\alpha}{3\sqrt{\left(\frac{1}{9}(M - L_p)^2 + (M - T)^2\right)\alpha}} \quad (10)$$

$$= \frac{[m_1 - lsl_3 + (m_2 - lsl_2 - m_1 + lsl_3)\alpha, m_3 - lsl_1 + (m_2 - lsl_2 - m_3 + lsl_1)\alpha]}{3\sqrt{z_1 + (z_2 - z_1)\alpha}, 3\sqrt{z_3 + (z_2 - z_3)\alpha}}$$

$$= \left[\frac{m_1 - lsl_3 + (m_2 - lsl_2 - m_1 + lsl_3)\alpha}{3\sqrt{z_3 + (z_2 - z_3)\alpha}}, \frac{m_3 - lsl_1 + (m_2 - lsl_2 - m_3 + lsl_1)\alpha}{3\sqrt{z_1 + (z_2 - z_1)\alpha}} \right]$$

where,

$$z_1 = \frac{1}{9}(m_1^2 - 2m_1lp_3 + lp_3^2) + m_1^2 - 2m_1t_3 + t_3^2$$

$$z_2 = \frac{1}{9}(m_2^2 - 2m_2lp_2 + lp_2^2) + m_2^2 - 2m_2t_2 + t_2^2$$

$$z_3 = \frac{1}{9}(m_3^2 - 2m_3lp_1 + lp_1^2) + m_3^2 - 2m_3t_1 + t_1^2$$

$$C_{pmk}(\alpha) = \min(C_{pmu}(\alpha), C_{pml}(\alpha)) \quad (11)$$

In the previous equations, USL_α , LSL_α , and T_α denote the fuzzy upper specification limit, fuzzy lower specification limit, and fuzzy target value of the quality characteristic, respectively. In addition, $L_{p\alpha}$ denotes fuzzy percentile $x_{0.135}$, M_α represents fuzzy percentile x_{50} , and $U_{p\alpha}$ corresponds to fuzzy percentile $x_{99.865}$ of the process data. These variables are expressed in terms of TFNs through the following equations:

$$USL_\alpha = [usl_1 + (usl_2 - usl_1)\alpha, usl_3 + (usl_2 - usl_3)\alpha] \quad (12)$$

$$LSL_\alpha = [lsl_1 + (lsl_2 - lsl_1)\alpha, lsl_3 + (lsl_2 - lsl_3)\alpha] \quad (13)$$

$$T_\alpha = [t_1 + (t_2 - t_1)\alpha, t_3 + (t_2 - t_3)\alpha] \quad (14)$$

$$L_{p\alpha} = [lp_1 + (lp_2 - lp_1)\alpha, lp_3 + (lp_2 - lp_3)\alpha] \quad (15)$$

$$M_\alpha = [m_1 + (m_2 - m_1)\alpha, m_3 + (m_2 - m_3)\alpha] \quad (16)$$

$$U_{p\alpha} = [up_1 + (up_2 - up_1)\alpha, up_3 + (up_2 - up_3)\alpha] \quad (17)$$

The methodology proposed in this study focuses on two main aspects: (1) proposing an optimal construction of TFNs associated with M , L_p , and U_p , and (2) applying a modified Clements method (Yen & Reza Langari, 1999) using metaheuristic techniques for parameter estimation. The entire methodology comprises five key steps, which are described below and illustrated in Figure 2.

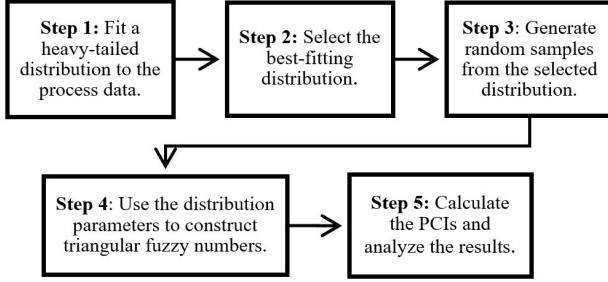


Fig. 2. Steps of the methodology

Step 1: Given n observations from a process with asymmetric data, fit a distribution that captures this behavior, enabling a proper analysis of the process quality performance. Typically, a heavy-tailed distribution – such as the Birnbaum–Saunders or Generalized Extreme Value – is selected, and its parameters are estimated using heuristic techniques (Carmona, 2014).

Step 2: If two or more distributions fit the process data, the Bayesian Information Criterion (BIC) is employed to select the best model (Schwarz, 1978). The parameters of the chosen distribution are then estimated using the Modified Particle Swarm Optimization (MPSO) technique (Yolcu et al., 2013), with parameters tuned according to the guidelines proposed by Pedersen (2010). Specifically, the swarm size was set to $25 + 1.4N_f$ (N_f is the number of free parameters), cognitive and social coefficients to $c_1 = 2.8$ and $c_2 = 1.3$, and a constriction factor $k_0 = 1$. Search bounds for the parameters were defined based on prior observations and domain knowledge of the distribution.

Step 3: Based on the fitted distribution model and the parameters estimated in Step 2, m random samples of size n are generated.

Step 4: TFNs are constructed using the parameters of the selected distribution, with a 95% confidence interval (Arteaga Sierra et al., 2017). For each random sample generated in Step 3, percentiles $x_{0.135}$, x_{50} , and $x_{99.865}$ are calculated. From these, vectors M , U_p , and L_p are obtained, and their respective means and variances ($\bar{M}$, $\bar{L}$, $\bar{U}$, s_M , s_L , and s_U) are computed.

The TFNs are determined as follows:

$$M_\alpha = \left(\bar{M} - 1.96 \frac{s_M}{\sqrt{n}}, \bar{M}, \bar{M} + 1.96 \frac{s_M}{\sqrt{n}} \right) \quad (18)$$

$$= (M_l, M_r)$$

$$L_{p\alpha} = \left(\bar{L} - 1.96 \frac{s_L}{\sqrt{n}}, \bar{L}, \bar{L} + 1.96 \frac{s_L}{\sqrt{n}} \right) \quad (19)$$

$$= (L_{pl}, L_{pr})$$

$$U_{p\alpha} = \left(\bar{U} - 1.96 \frac{s_U}{\sqrt{n}}, \bar{U}, \bar{U} + 1.96 \frac{s_U}{\sqrt{n}} \right) \quad (20)$$

$$= (U_{pl}, U_{pr})$$

Step 5: Finally, using the fuzzy specification limits with asymmetric tolerances, the PCIs are estimated based on (9), (10), and (11). The fuzzy index obtained is referred to as HFS-PCI, and the results are then analyzed and interpreted.

Results and Discussion

The proposed methodology was implemented in Python to facilitate calculations and simulations. Data from a case study in the electronics industry (Tab. 1), sourced from Leiva et al. (2014) and Hsu et al. (2008), were used. In the electronics industry, assembling integrated circuits involves several critical stages, including wafer cutting, wire bonding, and component encapsulation. Among these, the diameter of the solder balls is a key parameter that directly impacts the quality of electrical connections.

This study aims to evaluate process capability in wire bonding within integrated circuit manufacturing, focusing specifically on the diameter of solder balls as the primary quality characteristic. The employed data exhibit a nonnormal distribution due to their asymmetry and feature asymmetric fuzzy tolerances. The results for each stage of the applied methodology are presented below.

Step 1: A heavy-tailed distribution was fitted to the solder ball diameter data shown in Table 1. The results of the Kolmogorov–Smirnov tests for distribution fit, presented in Table 2, indicate that the data do not follow a normal distribution.

Step 2: The distribution fits were assessed using the Bayesian Information Criterion (BIC), which identified the Weibull distribution as the best fit for the data. The shape (k), scale (λ), and location (μ) parameters for the Weibull distribution are shown in Table 3. Additionally, Figure 3 presents the probability density function for the Weibull distribution based on the parameters estimated via the MPSO.

Table 1
Solder ball diameter data (mm)

2.891	4.035	4.495	2.890	2.312
3.842	1.590	1.954	1.842	0.68
0.619	2.788	1.050	3.750	3.508
4.805	1.516	2.227	2.797	1.636
1.770	3.492	5.706	3.722	6.644
2.111	3.591	2.093	3.222	2.891
1.508	1.811	2.803	6.659	0.923
2.495	0.921	4.061	9.725	1.600
3.696	1.982	2.413	5.480	1.992
4.882	1.380	3.982	2.260	2.366
3.158	5.228	3.334	5.896	5.639
2.752	1.301	2.260	0.889	2.381
6.123	6.549	5.954	2.207	4.417
1.066	0.94	4.101	4.542	1.295
2.472	1.383	4.494	1.694	2.892
2.582	0.665	3.234	1.102	1.083
6.229	3.177	2.333	1.311	4.419
4.281	3.360	1.131	1.618	4.489
2.573	1.845	4.620	6.221	1.694
2.899	3.782	2.336	1.175	3.055

Table 2
Probability distribution fits for process data

Distribution	p-value
Weibull	0.9997
Birnbaum–Saunders	0.9902
Log-normal	0.9849
Generalized extreme value	0.9737
Generalized Pareto	0.6195
Student's t	0.4408
Logistic	0.4238
Normal	0.1509
Pareto	0.0569
Cauchy	0.0219

Steps 3 and 4: From the Weibull distribution, 1,000 samples of size 200 were generated using the estimated parameters. TFNs (M_α , $L_{p\alpha}$, and $U_{p\alpha}$) were then calculated by applying (18) to (20).

Step 5: Given the specification limits set for the solder ball diameter ($LSL = 0.5$ mm and $USL = 8.0$ mm), an asymmetric target specification (T) of 2.2 mm was established. This target aligns with the process

Table 3
Estimated parameters for the Weibull distribution

Type of parameter	Value
Shape ($\hat{k}$)	1.4376
Scale ($\hat{\lambda}$)	0.5725
Location ($\hat{\mu}$)	2.7079

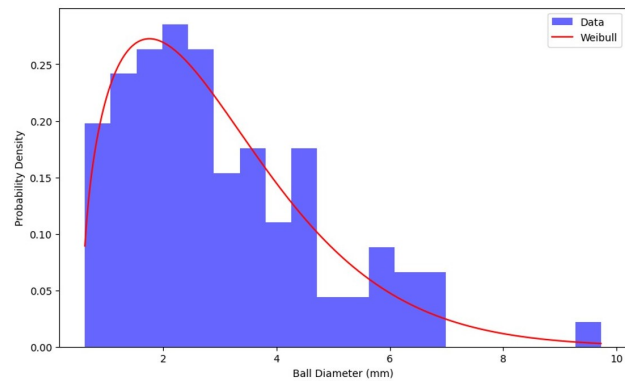


Fig. 3. Distribution fit for the solder ball diameter data (mm)

data's central tendency, ensuring a suitable diameter that minimizes failure risks and maintains the quality of electrical connections.

Following the methodology proposed by [Senvar & Kahraman \(2014\)](#), TFNs for the fuzzy specification limits were then constructed, and the fuzzy index $C_{pmk}(\alpha)$ was computed using (9), (10), and (11) (Tab. 4).

Table 4
Estimated fuzzy PCIs

α	$C_{pml}(\alpha)$		$C_{pmu}(\alpha)$	
0.0	0.696	1.260	1.235	1.968
0.1	0.713	1.214	1.259	1.911
0.2	0.732	1.171	1.284	1.859
0.3	0.751	1.132	1.310	1.810
0.4	0.771	1.095	1.338	1.764
0.5	0.792	1.061	1.367	1.721
0.6	0.814	1.028	1.397	1.680
0.7	0.838	0.998	1.429	1.641
0.8	0.863	0.970	1.463	1.604
0.8	0.889	0.943	1.498	1.569
1.0	0.918	0.918	1.536	1.536

A comparison between the fuzzy indices $C_{pml}(\alpha)$ and $C_{pmu}(\alpha)$, the traditional normality-index (using equation (6)) and Clements (using equation (7)) is

shown in Figure 4. The results indicate that a fuzzy approach, which simultaneously deals with asymmetry and non-normality, can yield PCIs aligned with the specific data and process characteristics. It can be observed that the proposed method allows for a more accurate point estimate because it better captures the true characteristics of the process data and their uncertainty, thereby reducing the risk of underestimating the process capability if traditional methods were used to analyze it.

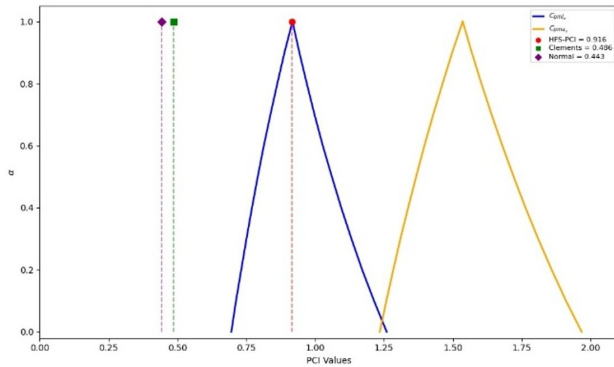


Fig. 4. Comparative results of $C_{pml}(\alpha)$, $C_{pmu}(\alpha)$, and C_{pmk}

In order to validate the performance of the proposed methodology, several robustness tests were conducted to assess its accuracy and precision under various extreme scenarios. Initially, taking a target of the HFS-PCI index, i.e., $C_{pmk}(1) = C_{pml}(1) = 0.916$, errors of the traditional methods relative to the proposed fuzzy index were calculated (Table 5). The results indicate that traditional PCIs often underestimate process capability when the assumption of data normality is not met, and tolerances are asymmetric. By addressing these issues, the proposed fuzzy approach enhances process capability analysis, offering a more accurate reflection of actual process conditions.

Table 5
Estimated fuzzy PCIs

Method	Estimated value	Relative Error (%)
HFS-PCI	0.916	–
Clements	0.486	–46.94
Normal	0.443	–51.64

In the following test, an analysis was conducted of the accuracy of the HFS-PCI index compared to traditional methods, when the true value of C_{pmk} is known. To this end, a simulation scheme analogous to that proposed by Ahmad et al. (2007) was followed.

The value $C_{pmk} = 1.0$ was set, and based on the percentiles $x_{0.135}$, x_{50} , and $x_{99.865}$, $T = 2.2$, and $USL = 8.2$, the appropriate value of $LSL = 0.2$ was determined using Eq. (21).

$$LSL = x_{50} - C_{pmk} \left(3 \sqrt{\left(\frac{x_{99.865} - x_{0.135}}{6} \right)^2 + (x_{50} - T)^2} \right) \quad (21)$$

Next, 30 samples of size 100 were simulated from a Weibull distribution with the parameter values given in Table 3 and estimates of C_{pmk} were calculated for each sample using the steps of each method. For the HSF-PCI, $C_{pmk}(1)$ was taken as its estimate. The optimal method is defined by its accuracy and precision; that is, the one whose average estimates converge toward the true value and exhibit the lowest variability across replicates. Figure 5 shows the box plots for the three evaluated methods. The results reveal that the proposed HSF-PCI index not only remains closer to the reference value $C_{pmk} = 1.0$ but also exhibits significantly lower dispersion compared to both the Clements method and the traditional approach under the normality assumption.

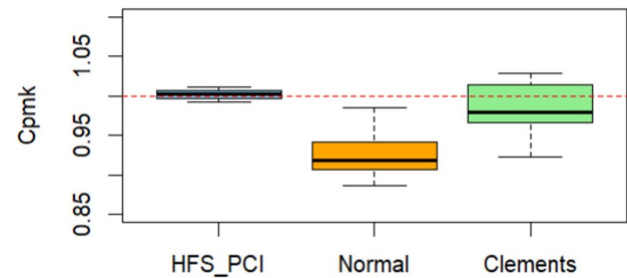


Fig. 5. Box plot of estimated C_{pmk} with target $C_{pmk} = 1.0$ (red line)

In addition, a sensitivity analysis was performed to assess the robustness of the estimates to changes in the data distribution. To this end, 25 types of Weibull distributions were considered, based on combinations of the shape parameter values, $k = (0.9376, 1.2376, 1.4376, 1.6376, 1.9376)$ and the scale parameter, $\lambda = (0.0725, 0.3725, 0.5725, 0.7725, 1.0725)$. For each of the distributions, 1,000 samples of size 100 were simulated, and C_{pmk} estimates were obtained using the three methods, within the specification limits detailed in step 5 of the methodology. Figure 6 shows the mean value of the estimates using the HFS-PCI method for each of the 25 distributions. It is verified that the model is robust for estimating the index even in the face of significant changes in the parameters of the Weibull distribution.

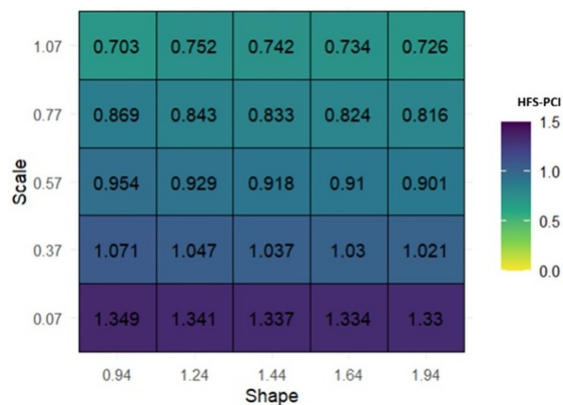


Fig. 6. Heatmap of average HFS-PCI index values across 25 scenarios

Finally, the variability of the estimates obtained in the runs for each method was evaluated across all 25 scenarios (Fig. 7). The results indicate that the proposed HFS-PCI index exhibits high internal consistency, maintaining minimal dispersion in the estimates across the 1,000 replicates performed for each scenario. This ensures stable estimates regardless of the inherent randomness in each simulation.

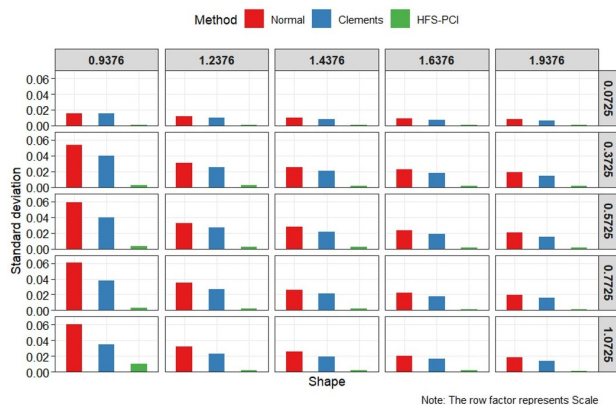


Fig. 7. Comparison of the standard deviation of C_{pmk} estimates for the three evaluated methods across 25 distribution scenarios

Conclusions

This study addressed the challenge of PCI estimation when data are non-normally distributed, and tolerances are both asymmetric and uncertain. Traditional PCI estimation methods are unsuitable in such cases, as they often misrepresent process capability by either overestimating or underestimating it. By applying a fuzzy logic approach, we demonstrated that adjusting for fuzzy and asymmetric specification limits more accurately captures actual process variability.

This study provides a novel integrated framework combining non-normal statistical modeling, fuzzy uncertainty representation, and metaheuristic optimization for process capability analysis under asymmetric tolerances, validated through comparative and simulation-based analysis.

The findings reveal a significant difference between capability indices calculated through traditional methods and those obtained by incorporating uncertainty and asymmetric tolerances. Specifically, traditional C_{pk} estimation produces a different and smaller confidence interval than that achieved with $C_{pmk\alpha}$ using the proposed methodology. This difference is attributed to the fact that the proposed approach more accurately addresses uncertainty and the process asymmetric nature, effectively overcoming the limitations of conventional methods and providing a more precise analysis.

This work has developed an innovative methodology for estimating Process Capability Indices (PCIs) in the context of non-normal data and asymmetric, imprecise tolerances. Consequently, future research could focus on several directions to extend and validate the applicability of this methodology in different industrial sectors, such as automotive, pharmaceutical, and energy.

In addition, it would be relevant to optimize the metaheuristic methods used in this study to improve the precision of parameter estimation, which could further refine the accuracy of capability indices under conditions of uncertainty. The application of advanced techniques, such as AI-based optimization, could facilitate data processing and adaptation in systems with high variability.

Future research could include the development of real-time applications of fuzzy PCIs for dynamic monitoring in automated quality control systems. Integrating this methodology into an automated environment would allow for rapid adaptation to changes in production while maintaining consistent quality standards. This would not only strengthen the robustness of the control system, but also align the proposed methodology with current industry demands for flexibility and responsiveness to variations in manufacturing processes.

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